Properties of large Lotka Volterra systems with random interactions

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joint work with

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Lotka-Volterra systems of coupled differential equations

A popular model to describe the dynamics of interacting species in foodwebs is given by a system of Lotka-Volterra equations:

$$\boxed{\frac{dx_k(t)}{dt} = x_k(r_k - x_k + (B\boldsymbol{x})_k)}\,, \quad k \in [n]\,, \quad \boldsymbol{x} = (x_k)\,.$$

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1. if $\boldsymbol{x}|_{t=0} > 0$ then for all $t > 0$, $\boldsymbol{x}(t) > 0$.
2. if $B = 0$ (no interactions), we recover the logistic equation

$$\frac{dx_k(t)}{dt} = x_k(r_k - x_k).$$

Questions

- **Existence of an equilibrium $\boldsymbol{x}^* = (x_k^*)$** such that

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- **Feasibility** of this equilibrium: $x_k^* > 0 \quad \text{for all} \quad k \in [n]$
- **Species extinction** $x_k^* = 0 \quad \text{for some} \quad k \in [n]$? In the latter case, we have

$$\begin{cases} \text{surviving species if} \quad x_k^* > 0\,, \\ \text{vanishing species if} \quad x_k^* = 0\,. \end{cases}$$

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- **The elliptic model:** encodes the natural correlation between $B_{k\ell}$ and $B_{\ell k}$ but limited because of a unique single trend

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- **Sparse models:** encodes the fact that a species only interacts with $d \ll n$ other species.

Assumption 2: $n \to \infty$

This assuption is relevant

- to model large foodwebs with many species
- to take advantage of self-averaging properties of large random matrices
- and leverage on random matrix theory

We need to normalize accordingly the interaction matrix so that (for instance)

$$\|B\| = \|B_n\| = \mathcal{O}(1)$$

as $n \to \infty$.

Equilibrium and global stability

Theorem (Takeuchi & Adachi 1980)

Consider the LV system

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If there exists a diagonal positive matrix W such that

$$W(-I + B) + (-I + B^T)W \;<\; 0 \quad \text{(negative definite)}$$

then if $\boldsymbol{x}|_{t=0} > 0$, system (1) has a unique non negative stable equilibrium:

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Remark on uniqueness

- if $\boldsymbol{x}|_{t=0} > 0$ then $\boldsymbol{x}^*$ is the unique solution of the **Linear Complementarity Problem** (LCP):

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- if $x_1|_{t=0} = 0$, just consider the subsystem where x_1's interactions are erased in matrix B.

Equilibrium and global stability: i.i.d. model

Corollary I (RMT - i.i.d. case)

Assume that $B_{k\ell} = \frac{A_{k\ell}}{\alpha\sqrt{n}}$ where $\begin{cases} A_{k\ell} \text{ i.i.d.}\,, \\ \mathbb{E}A_{k\ell} = 0\,, \\ \mathbb{E}A_{k\ell}^2 = 1 \end{cases} \quad + \quad \mathbb{E}|A_{k\ell}|^4 < \infty\,.$

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Well-known that $\lambda_{\max}\left(\frac{A+A^T}{\sqrt{2}\sqrt{n}}\right) \xrightarrow[n\to\infty]{a.s.} 2$ - we conclude easily.

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- The choice $W = I$ might not be optimal.

Equilibrium and global stability: elliptic model I

Let $A = (A_{ij})$ a $n \times n$ matrix. Assume that

- The (A_{ii}) are i.i.d $\mathcal{N}(0,1)$, the (A_{ij}, A_{ji}) are i.i.d. $\mathcal{N}_2\left(0, \begin{pmatrix} 1, \rho \\ \rho, 1 \end{pmatrix}\right)$
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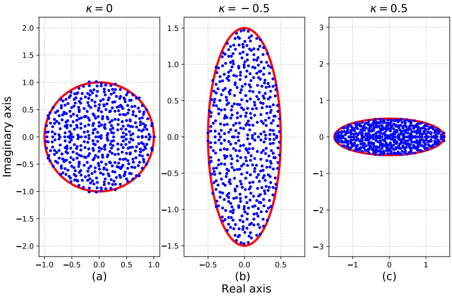

Figure: Centered elliptical model ($\mu = 0$) for various correlations ρ. Notice that $\rho = 0$ represents the model with i.i.d. entries.

Equilibrium and global stability: elliptic model II

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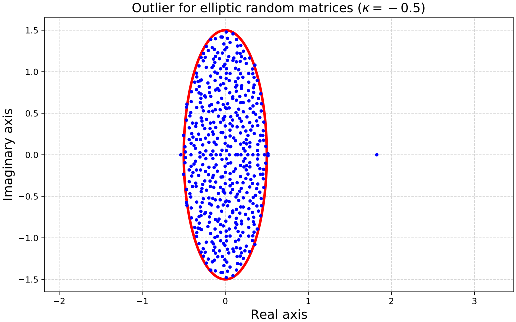

Figure: Elliptic model with $\mu = 2$. The outlier is very close to μ.

Equilibrium and global stability: elliptic model III

Corollary II (RMT - elliptic case)

Consider the following set of **admissible parameters**:

$$\mathcal{A} = \Bigg\{(\rho, \alpha, \mu) \in (-1, 1) \times (0, \infty) \times \mathbb{R}\,, \quad \alpha > \sqrt{2(1+\rho)}, \quad \mu < \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{2(1+\rho)}{\alpha^2}}\Bigg\}$$

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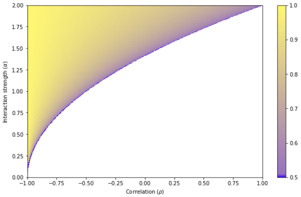

Figure: Representation of the set of admissible parameters $\mathcal{A}$ by a heat map. The x-axis corresponds to ρ, the y-axis to σ and the intensity of the color μ.

Open question

Statistical properties of the equilibrium

Consider the i.i.d. model and $\alpha > \sqrt{2}$. The equilibrium $\boldsymbol{x}^*$ is the solution of the LCP problem

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- $+$ other statistical properties of equilibrium $\boldsymbol{x}^*$?
- Yes, using statistical physics techniques, but no mathematical proof so far.

Reference

- Ecological communities with Lotka-Volterra dynamics, G. Bunin, Phys. Rev. E (2017)

Feasible equilibrium: a simple linear equation

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- If matrix $I - B$ is invertible, then

$$\boldsymbol{x}^* = (I - B)^{-1}\boldsymbol{r}\,.$$

No feasible equilibrium under standard normalization

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and $A_{k\ell} \sim \mathcal{N}(0,1)$ i.i.d., **there is no feasible equilibrium with proba 1**

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References

- "The feasibility of equilibria in large ecosystems: A primary but neglected concept in the complexity-stability debate",
  Dougoud, Vikenbosch, Rohr, Bersier, Mazza, PLoS Comput. Biology, **2018**
- "A chaos hypothesis for some large systems of random equations". Geman and Hwang, **1982**.

Elements of proof

Theorem (Geman, Hwang)

- Let M fixed, $\boldsymbol{\alpha} > 4$ and $\boldsymbol{x}^* = \mathbf{1} + \frac{A}{\alpha\sqrt{n}}\boldsymbol{x}^*$

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- If $\boldsymbol{\alpha} > 4$ fixed, the probability to obtain a positive solution goes to zero:

$$\mathbb{P}\left\{\inf_{k\in[n]} x_k^* > 0\right\} \leq \mathbb{P}\left\{\inf_{k\in[M]} x_k^* > 0\right\} \sim \prod_{k\in[M]} \mathbb{P}\left\{x_k^* > 0\right\} \xrightarrow[M\to\infty]{} 0\,.$$

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Conclusion

- Feasible solutions for $\boxed{\boldsymbol{x}^* = \mathbf{1} + \frac{A}{\boldsymbol{\alpha}\sqrt{N}}\boldsymbol{x}^*}$ are eventually extremely rare.

Feasibility of the solution

Consider the system

$$\boldsymbol{x}^* = \mathbf{1} + \frac{A}{\boldsymbol{\alpha}\sqrt{n}}\boldsymbol{x}^* \quad \text{where} \quad \boldsymbol{\alpha} = \boldsymbol{\alpha}_n \xrightarrow[n\to\infty]{} \infty\,.$$

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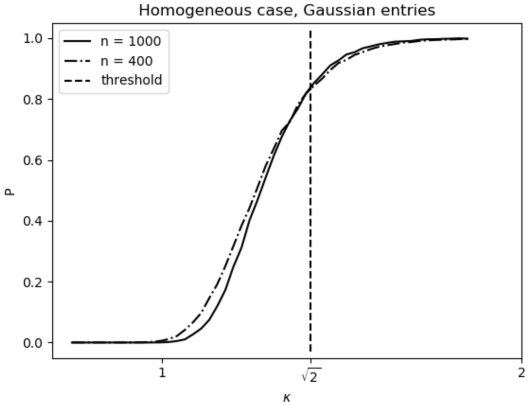
References

- Positive solutions for large random linear systems, Bizeul-N., Proc AMS, **2021**

Phase transition (gaussian case)



- We plot the frequency of positive solutions over 10000 trials for the system

$$\boldsymbol{x}^* = \mathbf{1} + \frac{1}{\kappa\sqrt{\log(n)}} \frac{A}{\sqrt{n}} \boldsymbol{x}^*$$

as a function of the parameter κ.

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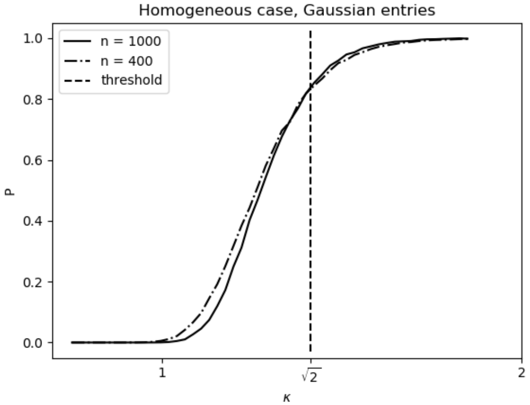

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- A phase transition occurs at the critical value $\kappa = \sqrt{2}$.

Important facts

Gaussian extreme values

- Let $(Z_k)_{k\in[n]}$ i.i.d. $\mathcal{N}(0,1)$ random variables, Denote by

$$\check{M}_n = \min_{k\in[n]} Z_k$$

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Existence of the resolvent

- Recall that

$$\rho\left(\frac{A}{\sqrt{n}}\right) \xrightarrow[n\to\infty]{a.s.} 1 \qquad \text{and} \qquad \left\|\frac{A}{\sqrt{n}}\right\| \xrightarrow[n\to\infty]{a.s.} 2\,.$$

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As a consequence, if $\boldsymbol{\alpha} > 1$ then $\left(I - \frac{A}{\alpha\sqrt{n}}\right)$ is eventually invertible and

$$\boxed{\boldsymbol{x}^* = \left(I - \frac{A}{\boldsymbol{\alpha}\sqrt{n}}\right)^{-1}\mathbf{1}}$$

is well-defined.

A heuristics for the critical scaling

1. **Unfold the resolvent**.

$$x_k^* \quad = \quad \left[\left(I - \frac{A}{\alpha\sqrt{n}} \right)^{-1} \mathbf{1} \right]_k$$

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Crux of proof: to handle the remaining term R_k

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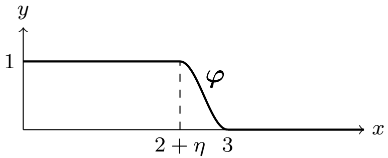
$\Rightarrow$ The main effort is to prove that $A \mapsto \widetilde{R}_k(A)$ is K-Lipschitz.

A truncated version of the remainder term

- Let $\varphi : \mathbb{R}^+ \to [0, 1]$ a smooth **cut-off** function



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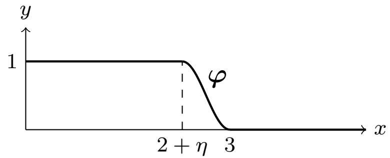

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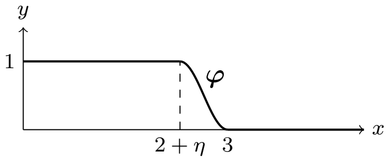

- Recall that $\left\| \frac{A}{\sqrt{n}} \right\| \xrightarrow[n\to\infty]{a.s.} 2$, consider $\boxed{\tilde{R}_k = \varphi\left(\left\| \frac{A}{\sqrt{n}} \right\| \right) R_k}$

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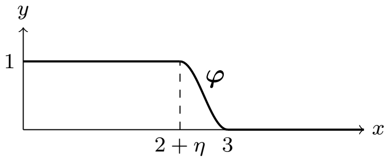

- Recall that $\left\| \frac{A}{\sqrt{n}} \right\| \xrightarrow[n\to\infty]{a.s.} 2,$ consider $\boxed{\tilde{R}_k = \varphi\left(\left\| \frac{A}{\sqrt{n}} \right\|\right) R_k}$
- Notice that

$$\mathbb{P}\left(\max_{k\in[n]} R_k \neq \max_{k\in[N]} \tilde{R}_k\right) \\ \leq \mathbb{P}(\exists k_0,\ R_{k_0} \neq \tilde{R}_{k_0}) = \mathbb{P}\left(\left\|A/\sqrt{n}\right\| \geq 2+\eta\right) \xrightarrow[n\to\infty]{} 0\,.$$

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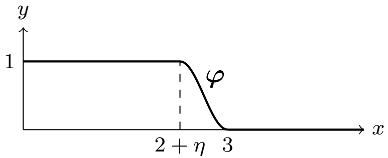

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- No asymptotic loss when replacing R_k by $\widetilde{R}_k$.

Proof of sub-gaussianity of $\tilde{R}_k$: concentration

- We first prove that $A \mapsto \tilde{R}_k(A)$ is K-lipschitz:

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- We then proceed by density to complete the proof of the Lipschitz property.

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References

- Explorability and the origin of network sparsity in living systems, by Busiello et al. *Scientific reports*, **2017**.
- Feasibility of sparse large Lotka-Volterra ecosystems, by Akjouj and N., **2021**.

More on the block matrix structure assumption

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Open question

- Possible to relax this **block structure assumption**? Simulations suggest yes.

Feasibility for the elliptical model

Theorem (Clenet, El Ferchichi, N.)

Consider the model

$$B(\boldsymbol{\alpha}) = \frac{A}{\boldsymbol{\alpha}\sqrt{n}} + \frac{\mu}{n}\mathbf{1}\mathbf{1}^T \ ,$$

and assume that $\mu < 1$. Then the same phase transition as before occurs.

Non-homogeneous case I

Let $\boldsymbol{r}$ is $N \times 1$ deterministic. We are interested in the equation

$$\boxed{\boldsymbol{x} = \boldsymbol{r} + \frac{A}{\boldsymbol{\alpha}\sqrt{N}}\boldsymbol{x}} \quad \text{where} \quad \begin{cases} \boldsymbol{r}_{\min}(n) = \min_k r_k \\ \boldsymbol{r}_{\max}(n) = \max_k r_k \\ \sigma_{\boldsymbol{r}}(n) = \sqrt{\frac{1}{N}\sum_k r_k^2} \end{cases}$$

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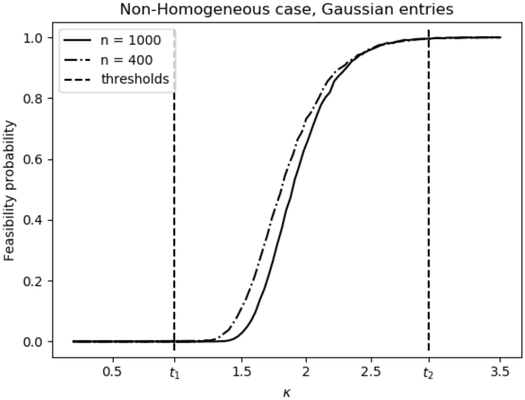
- if $\frac{\boldsymbol{\alpha}_N}{\boldsymbol{\alpha}_N^*} \geq (1+\delta)\frac{\sigma_{\boldsymbol{r}}(n)}{\boldsymbol{r}_{\min}(n)}$ then $\boxed{\mathbb{P}\left\{\inf_{k\in[N]} x_k > 0\right\} \xrightarrow[N\to\infty]{} 1}$.

Non-homogeneous case II



- In the non-homogeneous case, there is a transition buffer

$$\frac{\boldsymbol{\alpha}_N}{\boldsymbol{\alpha}_N^*} \in \left[\frac{\sigma_{\boldsymbol{r}}(n)}{\boldsymbol{r}_{\max}(n)}, \frac{\sigma_{\boldsymbol{r}}(n)}{\boldsymbol{r}_{\min}(n)}\right]$$

and not a sharp transition at $\frac{\boldsymbol{\alpha}_N}{\boldsymbol{\alpha}_N^*} \sim 1$.

Thank you for your attention!