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Random G-circulant matrices. Spectrum of random convolution operators on large finite groups

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Framework & notation

- main object: an $N \times N$ matrix $A = A_N$ with random coefficients, $N \to \infty$
- eigenvalues $\lambda_i = \lambda_i(A) \in \mathbb{C}$
- (empirical) spectral measure:

$$L_A = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}, \quad L_A(I) = \frac{1}{N} |\{i \colon \lambda_i \in I\}|,$$

- average spectral measure

$$\overline{L}_A = \mathbb{E} L_A$$

- L_A – „global behaviour" of eigenvalues
- $L_{\sqrt{AA^*}}$ – global behaviour of singular values
- **Main question:** Do $L_A, L_{\sqrt{AA^*}}$ converge as $N \to \infty$?

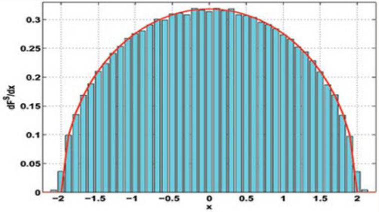
Classical results (I): semicircular law

- $A_N = \frac{1}{\sqrt{N}}[X_{ij}]_{i,j=1}^N$, $X_{ij} = \overline{X}_{ji}$
- X_{ij}, $i \leqslant j$ – indep., X_{ij}, $i < j$ – i.i.d. and X_{ii} – i.i.d.
- $\mathbb{E}X_{ij} = 0$, $\mathbb{E}|X_{ij}|^2 = 1$.

Theorem (Wigner + ...)

With probability one, L_{A_N} converges weakly to the measure ρ_{sc} given by density

$$\frac{d\rho_{sc}}{dx} = \frac{1}{2\pi}\sqrt{4 - x^2}\mathbb{1}_{[-2,2]}(x)$$



Pic. Alan Edelman

Classical results (II): quartercircle law

- $A_N = \frac{1}{\sqrt{N}}[X_{ij}]_{i,j=1}^{N}$
- X_{ij} – independent, identically distributed
- $\mathbb{E}X_{ij} = 0$, $\mathbb{E}|X_{ij}|^2 = 1$.

Theorem (Marchenko–Pastur)

With probability one $L_{\sqrt{A_N A_N^*}}$ converges weakly to the measure ρ_∞ given by density

$$\frac{d\rho_\infty}{dx} = \frac{1}{\pi}\sqrt{4 - x^2}\,\mathbb{1}_{[0,2]}(x)$$

Classical results (II): quartercircle law

If X_{ij} – standard complex Gaussian variables

$$X_{ij} = \frac{1}{\sqrt{2}}(g_{ij} + \sqrt{-1}g_{ij'}), \quad g_{i,j}, g'_{ij} \text{ – i.i.d. } N(0,1),$$

and $\rho_N = \overline{L}_{\sqrt{A_N A_N^*}}$, then on $\mathbb{R}_+$,

$$\frac{d\rho_N}{dx} = 2xe^{-Nx^2} \sum_{\ell=0}^{N-1} (\mathcal{L}_\ell(Nx^2))^2,$$

where $\mathcal{L}_\ell$ – Laguerre's polynomials

$$\mathcal{L}_\ell(x) = \frac{e^x}{\ell!} \frac{d^\ell}{dx^\ell}(e^{-x}x^\ell) = \sum_{k=0}^{\ell} \binom{\ell}{k} \frac{(-1)^k}{k!} x^k$$

Classical results (III): circular law

- $A_N = \frac{1}{\sqrt{N}}[X_{ij}]_{i,j=1}^N$
- X_{ij} – independent, identically distributed
- $\mathbb{E}X_{ij} = 0$, $\mathbb{E}|X_{ij}|^2 = 1$.

Theorem (Mehta, Girko, . . . , Tao–Vu)

With probability one L_{A_N} converges weakly to the uniform distribution on the unit disc in $\mathbb{C}$ (denoted by θ_∞).

Theorem (Ginibre–Mehta)

If X_{ij} – standard complex Gaussians, $\theta_N = \overline{L}_{A_N}$, then

$$\frac{d\theta_N(z)}{dz} = \frac{1}{\pi} e^{-N|z|^2} \sum_{\ell=0}^{N-1} \frac{N^\ell |z|^{2\ell}}{\ell!}.$$

Matrices with additional structure

- In 1999 r. Z. Bai asked about the limiting behaviour of spectral measures of symmetric random matrices with additional linear structure, in particular Toeplitz matrices

$$T_N = \frac{1}{\sqrt{N}}[X_{|i-j|}]_{i,j=1}^N = \frac{1}{\sqrt{N}} \begin{bmatrix} X_0 & X_1 & \cdots & \cdots & X_{N-2} & X_{N-1} \\ X_1 & X_0 & X_1 & \cdots & X_{N-3} & X_{N-2} \\ X_2 & X_1 & X_0 & X_1 & \cdots & X_{N-3} \\ \vdots & \dots & \ddots & \ddots & \dots & \vdots \\ X_{N-2} & X_{N-3} & \ddots & \ddots & X_0 & X_1 \\ X_{N-1} & X_{N-2} & X_{N-3} & \cdots & X_1 & X_0 \end{bmatrix},$$

  where X_i – i.i.d., $\mathbb{E}X_i = 0$, $\mathbb{E}X_i^2 = 1$.
- Solved independently by Bryc, Dembo, Jiang and Hammond, Miller (2003).
- Many articles on matrices with additional structure in subsequent years (Bose, Chatterjee, Kargin, Massey, Meckes, R.A., Miller, Sen, Virág)

Matrices with additional structure

- For most models the limiting measure is known only through its moments
- One of the exceptions – circulant matrices

$$\mathcal{C}_N = \frac{1}{\sqrt{N}}[X_{i-j \bmod N}]_{i,j=1}^N = \frac{1}{\sqrt{N}} \begin{bmatrix} X_0 & X_{N-1} & \cdots & \cdots & X_2 & X_1 \\ X_1 & X_0 & X_{N-1} & \cdots & X_3 & X_2 \\ X_2 & X_1 & X_0 & X_{N-1} & \cdots & X_3 \\ \vdots & \cdots & \ddots & \ddots & \cdots & \vdots \\ X_{N-2} & X_{N-3} & \ddots & \ddots & X_0 & X_{N-1} \\ X_{N-1} & X_{N-2} & X_{N-3} & \cdots & X_1 & X_0 \end{bmatrix}$$

Theorem (Meckes)

If X_i are i.i.d., $\mathbb{E}X_i = 0$, $\mathbb{E}|X_i|^2 = 1$, then

- $L_{\mathcal{C}_N}$ converges weakly in probability to the standard Gaussian measure on $\mathbb{C}$
- $L_{\sqrt{\mathcal{C}_N \mathcal{C}_N^*}}$ converges to the measure with density $2xe^{-x^2}$ on $\mathbb{R}_+$.

Abelian G-circulants

- The matrix $\mathcal{C}_N$ describes convolution with the sequence $\mathbb{X}_N = N^{-1/2}(X_0, \ldots, X_{N-1})$ on the cyclic group $\mathbb{Z}_N$:
$$(\mathcal{C}_N x)(i) = (\mathbb{X}_N * x)(i) = \sum_{j \in \mathbb{Z}_N} \mathbb{X}_N(i-j)x(j)$$
- More generally, Meckes considered a sequence of Abelian groups G_N, $|G_N| \to \infty$ and random convolution operators
$$\mathcal{C}_N = \frac{1}{\sqrt{|G_N|}}[X_{hg^{-1}}]_{h,g \in G_N}.$$

Theorem (Meckes)

If X_g, $g \in G_N$ are i.i.d., $\mathbb{E}X_g = 0$, $\mathbb{E}X_g^2 = 0$, $\mathbb{E}|X_g|^2 = 1$, then

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- Meckes studied also the case $\mathbb{E}X_g^2 = \alpha \neq 0$ (in particular the case of real-valued X_g). The limit has a different form, one also needs additional assumptions on G_N.

Sketch of proof

- The matrices $\mathcal{C}_N$ are normal, so the case of singular values reduces to that of eigenvalues.
- All the matrices $\mathcal{C}_N$ are diagonal in the Fourier basis (the basis of characters on G_N), their eigenvalues are given by

$$\lambda_\varphi = \frac{1}{\sqrt{|G_N|}} \sum_{g \in G_N} X_g \overline{\varphi}(g), \quad \varphi \in \widehat{G}_N$$

- CLT shows that $\lambda_\varphi \xrightarrow{D} \theta_1$, moreover for every $f \colon \mathbb{C} \to [0,1]$, $f(\lambda_\varphi), f(\lambda_\psi)$ are asymptotically independent (we use orthogonality relations).
- Together with Chebyshev's inequality this gives

$$\int f(x) L_{\mathcal{C}_N}(dx) = \frac{1}{|G_N|} \sum_{\varphi \in \widehat{G}_N} f(\lambda_\varphi) \xrightarrow{\mathbb{P}} \int_{\mathbb{C}} f(z) d\theta_1(z)$$

General G-circulants

- Let G be a large (not necessarily Abelian) group, and

$$\mathcal{C} = \frac{1}{\sqrt{|G|}}[X_{hg^{-1}}]_{h,g\in G},$$

  where $X_g, g \in G$ are i.i.d., $\mathbb{E}X_g = \mathbb{E}X_g^2 = 0$, $\mathbb{E}|X_g|^2 = 1$.
- What can be said about the behaviour of eigenvalues/singular values?
- For which sequences of groups G_N, $|G_N| \to \infty$ do the measures $L_{C_N}, L_{\sqrt{C_N C_N^*}}$ converge weakly in probability?
- Does the limit depend on the algebraic structure of G_N?

Representation theory in one slide

- G – group, unitary representations – homomorphisms $\Lambda\colon G \to U(n)$, $n = \dim \Lambda$
- Λ – irreducible if there are no nontrivial invariant subspaces
- Λ_1, Λ_2 – equivalent if $\Lambda_1(g)F = F\Lambda_2(g)$ for some isomorphism $F\colon \mathbb{C}^n \to \mathbb{C}^n$
- $\widehat{G}$ – family of (equivalence classes) of irreducible representations of G
- $\{g \mapsto \Lambda_{ij}(g)\colon \Lambda \in \widehat{G}, i, j \leqslant \dim(\Lambda)\}$ – an orthogonal basis of $L_2(G)$, $\|\Lambda_{ij}\|_2 = \frac{1}{\sqrt{\dim \Lambda}}$
- Plancherel measure on $\widehat{G}$:

$$\mu_G(\{\Lambda\}) = \frac{(\dim \Lambda)^2}{|G|},\ \mu_G(\widehat{G}) = 1$$

- Projected Plancherel measure – measure on $\mathbb{Z}_+$ given by

$$\widetilde{\mu}_G(\{n\}) = \frac{n^2}{|G|}|\{\Lambda \in \widehat{G}\colon\ \dim \Lambda = n\}.$$

Fourier Transform

- Fourier transform of $x \in L_2(G)$:

$$\widehat{x}(\Lambda) = \sum_{g \in G} x_g \Lambda(g)$$

- The mapping $x \mapsto \widehat{x}$ is an isomorphism between $L_2(G)$ and $\bigoplus_{\Lambda \in \widehat{G}} \mathbb{C}^{\dim \Lambda} \otimes \mathbb{C}^{\dim \Lambda}$
- $\widehat{x * y}(\Lambda) = \widehat{x}(\Lambda)\widehat{y}(\Lambda)$
- Counterparts of the Plancherel formula, inversion formula, etc.

Representation theory – main consequences

- The convolution with $\mathbb{X}$ in the Fourier basis has a block-diagonal structure
- The blocks of size $\dim(\Lambda)^2 \times \dim(\Lambda)^2$ act on the subspaces $(\overline{\Lambda}_{ij})_{i,j\leqslant \dim(\Lambda)}$ through multiplication from the left by $\widehat{\mathbb{X}}(\Lambda)$
- The spectrum of left multiplication by $A \in \mathbb{C}^n \otimes \mathbb{C}^n$ is the same as the spectrum of A, the multiplicities of eigenvalue grow n times.

Proposition

If $\mathbb{X} = (X_g)_{g\in G}$, $\mathcal{C} = [X_{hg^{-1}}]_{h,g\in G}$, then the spectral measure of $\mathcal{C}$ (resp. $\sqrt{\mathcal{C}\mathcal{C}^*}$) is a mixture of spectral measures of the matrices $\widehat{\mathbb{X}}(\Lambda)$ (resp. $\sqrt{\widehat{\mathbb{X}}(\Lambda)\widehat{\mathbb{X}}(\Lambda)^*}$) driven by the Plancherel measure of the group:

$$L_{\mathcal{C}} = \sum_{\Lambda\in\widehat{G}} \mu_G(\Lambda) L_{\widehat{\mathbb{X}}(\Lambda)}.$$

Main result – singular values

Theorem (A.)

Assume that

- $|G_N| \to \infty$
- $\widetilde{\mu}_{G_N}$ converges weakly to a measure μ on $\mathbb{Z}_+ \cup \{\infty\}$
- ξ – random variable, $\mathbb{E}\xi = \mathbb{E}\xi^2 = 0$, $\mathbb{E}|\xi|^2 = 1$
- $\mathcal{C}_N = \frac{1}{\sqrt{|G_N|}}[X_{gh^{-1}}]_{g,h \in G_N}$, where X_g – independent copies ξ.

Then $L_{\sqrt{\mathcal{C}_N \mathcal{C}_N^*}}$ converges weakly in probability to the measure L_∞ with density

$$\frac{dL_\infty(x)}{dx} = \sum_{1 \leqslant n \leqslant \infty} \mu(n) \frac{d\rho_n(x)}{dx}.$$

In particular, if $\widetilde{\mu}_{G_N}$ converges to Dirac's delta at ∞, then L_∞ is the quartercirle law.

Sketch of proof - the Gaussian case

- If $X_g \sim \theta_1$, then $\widehat{\mathbb{X}}(\Lambda), \Lambda \in \widehat{G}$ are independent, moreover $\sqrt{\dim(\Lambda)}\widehat{\mathbb{X}}(\Lambda)_{ij}$ are i.i.d $\sim \theta_1$.
- Thus $\overline{L}_{\sqrt{\widehat{\mathbb{X}}(\Lambda)\widehat{\mathbb{X}}(\Lambda)^*}} = \mathbb{E}L_{\sqrt{\widehat{\mathbb{X}}(\Lambda)\widehat{\mathbb{X}}(\Lambda)^*}} = \rho_{\dim\Lambda}$
- Grouping the summands according to the dimension we get

$$L_{\sqrt{\mathcal{C}_N\mathcal{C}_N^*}} = \sum_n \frac{n^2}{|G_N|} \sum_{\dim(\Lambda)=n} L_{\sqrt{\widehat{\mathbb{X}}(\Lambda)\widehat{\mathbb{X}}(\Lambda)^*}}$$

$$\simeq \sum_{1\leqslant n\leqslant\infty} \widetilde{\mu}_{G_N}(n)\rho_n$$

The limiting behaviour based on two observations:

- For $n \to \infty$, $L_{\sqrt{\widehat{\mathbb{X}}(\Lambda)\widehat{\mathbb{X}}(\Lambda)^*}} \xrightarrow{\mathbb{P}} \rho_\infty$.
- For fixed n if $\tilde{\mu}_{G_N}(n) > \varepsilon > 0$, the LLN implies

$$\sum_{\dim(\Lambda)=n} L_{\sqrt{\widehat{\mathbb{X}}(\Lambda)\widehat{\mathbb{X}}(\Lambda)^*}} \simeq \rho_n |\{\Lambda \in \widehat{G}_N : \ \dim\Lambda = n\}|$$

Sketch of proof – the general case

- If X_g arbitrary, $\mathbb{E}X_g = \mathbb{E}X_g^2 = 0$, $\mathbb{E}|X_g|^2 = 1$ the contribution of low-dimensional irreps can be analyzed via CLT, as in the Abelian case.
- For high-dimensional irreps one shows that asymptotically (after some truncations)

$$\int x^{2k}\overline{L}_{\sqrt{\widehat{\mathbb{X}}(\Lambda)\widehat{\mathbb{X}}(\Lambda)^*}}(dx) = \mathbb{E}\frac{1}{\dim(\Lambda)}\mathrm{Tr}\Big(\frac{1}{|G_N|}\widehat{\mathbb{X}}(\Lambda)\widehat{\mathbb{X}}(\Lambda)^*\Big)^k$$

  depends only on $\mathbb{E}X_g = \mathbb{E}X_g^2 = 0$, $\mathbb{E}|X_g|^2 = 1$ (an easy combinatorial expansion)
- Thus for large N, $\overline{L}_{\sqrt{\widehat{\mathbb{X}}(\Lambda)\widehat{\mathbb{X}}(\Lambda)^*}} \simeq \rho_{\dim\Lambda} \simeq \rho_\infty$

Theorem (Talagrand)

If $Y_1, \ldots, Y_n$ are independent random variables with $\|Y_i\|_\infty \leqslant 1$, then for any convex 1-Lipschitz function $\varphi \colon \mathbb{R}^n \to \mathbb{R}$, and $t \geqslant 0$,

$$\mathbb{P}(|\varphi(Y_1, \ldots, Y_n) - \mathbb{E}\varphi(Y_1, \ldots, Y_n)| \geqslant t) \leqslant 2e^{-t^2/4}.$$

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Meckes and Szarek proved a concentration inequality for polynomials in random matrices, which in combination with Talagrand's theorem, after some truncations, gives

$$\int x^{2k} \overline{L}_{\sqrt{\widehat{\mathbb{X}}(\Lambda)\widehat{\mathbb{X}}(\Lambda)^*}}(dx) - \int x^{2k} L_{\sqrt{\widehat{\mathbb{X}}(\Lambda)\widehat{\mathbb{X}}(\Lambda)^*}}(dx) \xrightarrow{\mathbb{P}} 0.$$

Another approach to concentration and universality

Theorem (Polaczyk)

Assume that A_N is a sequence of $n_N \times n_N$ Hermitian random matrices, whose entries are uniformly square-integrable and can be partitioned into stochastically independent blocks of size at most m_N. Assume that d is any distance metrizing weak convergence of probability measures. If $n_N \to \infty$ and $m_N = o(n_N^2)$, then

$$d(L_{\frac{1}{\sqrt{n_N}} A_N}, \mathbb{E} L_{\frac{1}{\sqrt{n_N}} A_N}) \xrightarrow{\mathbb{P}} 0\,.$$

More recent approach

To deal with universality one should be also able to use recent results by Brailovskaya and van Handel.

Main result – eigenvalues

Theorem (A.)

Assume that

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- $\widetilde{\mu}_{G_N}$ converges weakly to a measure μ on $\mathbb{Z}_+ \cup \{\infty\}$
- $\mathcal{C}_N = \frac{1}{\sqrt{|G_N|}}[X_{gh^{-1}}]_{g,h\in G}$, where X_g – i.i.d standard complex Gaussian r.v.'s

Then $L_{\mathcal{C}_N}$ converges weakly in probability to the measure L_∞ with density

$$\frac{dL_\infty(x)}{dx} = \sum_{1\leqslant n\leqslant\infty} \mu(n)\frac{d\theta_n(x)}{dx}.$$

In particular, if $\widetilde{\mu}_{G_N}$ converges to Dirac's delta at ∞, then L_∞ is the uniform measure on the unit disc.

Examples

- G_N – Abelian. Then $\dim \Lambda = 1$ for $\Lambda \in \widehat{G}$, so limiting measures are ρ_1, θ_1 (as in Meckes' thm.)

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- F_q – a field with q elements, $G_q = GL_2(F)$. When $q \to \infty$, the limiting measures are again $\rho_\infty, \theta_\infty$

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- F_q – a field with q elements, $G_q = GL_2(F)$. When $q \to \infty$, the limiting measures are again $\rho_\infty, \theta_\infty$
- G – fixed, H_N–Abelian $|H_N| \to \infty$. Let $G_N = G \times H_N$. Limiting measures:

$$L_{\sqrt{\mathcal{C}_N \mathcal{C}_N^*}} \to \sum_n \widetilde{\mu}_G(n)\rho_n, \; L_{\mathcal{C}_N} \to \sum_n \widetilde{\mu}_G(n)\theta_n$$

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  E.g., for $G_N = S_3 \times \mathbb{Z}_N$

  $$L_{\sqrt{\mathcal{C}_N \mathcal{C}_N^*}} \to \frac{1}{3}\rho_1 + \frac{2}{3}\rho_2, \ L_{\mathcal{C}_N} \to \frac{1}{3}\theta_1 + \frac{2}{3}\theta_2$$

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- G – fixed, $G_N = G^{\times N}$. If G is Abelian in the limit one gets ρ_1, θ_1. If G is non-Abelian: $\rho_\infty, \theta_\infty$

A natural question

Which measures on $\mathbb{Z}_+ \cup \{\infty\}$ are weak limits of projected Plancherel measures of finite groups?

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Theorem (Czuroń)

- Assume that $\mu = p\delta_1 + (1-p)\delta_\infty$ for some $p \in [0,1]$. Then μ is a limit of projected Plancherel measures of a sequence of finite groups iff $p \in \{0\} \cup \{n^{-1} \colon n = 1, 2, 3, \ldots\}$.

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- For every positive integer n, the measure δ_n is a limit of projected Plancherel measures of a sequence of finite groups.

Other results

- If $\widetilde{\mu}_{G_N} \to \delta_\infty$, then families of independent random convolution operators are asymptotically free. This allows for computation of average spectral measures of polynomials in these operators.
- A CLT in the Gaussian case:

$$\frac{1}{\sqrt{|G_N|}}\Big(\mathrm{Tr}\, f\Big(\frac{1}{|G_N|}\mathcal{C}_N\mathcal{C}_N^*\Big) - \mathbb{E}\mathrm{Tr}\, f\Big(\frac{1}{|G_N|}\mathcal{C}_N\mathcal{C}_N^*\Big)\Big)$$

  for f of class C^1, Lipschitz.
  A different normalization then in classical CLT's for random matrices (e.g., Lytova–Pastur)

A few open problems

- Is there universality of the limiting spectral measure $L_{\mathcal{C}_N}$ (if G_N's have only irreps of bounded degree this follows from the approach by Meckes, in general requires analysis of the smallest singular value).
- Understand the local behaviour of eigenvalues.
- How does the operator norm behave?
- Extend results to the case $\mathbb{E}X_g^2 = \alpha \neq 0$.

Theorem (Meckes)

Let G_N be a sequence of Abelian groups with $|G_N| \to \infty$. Assume that X_g, $g \in G_N$ are i.i.d., $\mathbb{E}X_g = 0$, $\mathbb{E}X_g^2 = \alpha \in [0,1]$, $\mathbb{E}|X_g|^2 = 1$. Assume furthermore that there exists

$$p = \lim_{N\to\infty} \frac{|\{a \in G_N : a^2 = e\}|}{|G_N|}.$$

Then $L_{\mathcal{C}_N}$ converges weakly in probability to $(1-p)\gamma_0 + p\gamma_\alpha$, where γ_α is the centered Gaussian measure on $\mathbb{C} \simeq \mathbb{R}^2$ with covariance matrix

$$\frac{1}{2}\begin{bmatrix} 1+\alpha & 0 \\ 0 & 1-\alpha \end{bmatrix}.$$

Remarks:

- $\gamma_0 = \theta_1$
- $\alpha = 1$ corresponds to real-valued entries.

Real entries, non-Abelian groups

Define the Frobenius-Schur indicator of a representation

$$\iota(\Lambda) = \frac{1}{|G|} \sum_{g \in G} \mathrm{Tr} \Lambda(g^2) \in \{1, 0, -1\}$$

(corresponding to real, complex and quaternionic representations).

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(corresponding to real, complex and quaternionic representations). For $i \in \{-1, 0, 1\}$ define the measure $\widetilde{\mu}_G^{(i)}$ on $\mathbb{Z}_+$ as

$$\widetilde{\mu}_G^{(i)} = \frac{n^2}{|G|} |\{\Lambda \in \hat{G} \colon \dim \Lambda = n, \iota(\Lambda) = i\}|.$$

Theorem (Gerspach)

Assume that

- $|G_N| \to \infty$
- $\widetilde{\mu}^{(i)}_{G_N}$, $i = -1, 0, 1$, converge weakly to $\mu^{(i)}$ on $\mathbb{Z}_+ \cup \{\infty\}$
- ξ – a real random variable, $\mathbb{E}\xi = 0$, $\mathbb{E}\xi^2 = 1$
- $\mathcal{C}_N = \frac{1}{\sqrt{|G_N|}}[X_{gh^{-1}}]_{g,h \in G_N}$, where X_g – i.i.d. copies of ξ.

Then $L_{\sqrt{\mathcal{C}_N \mathcal{C}_N^*}}$ converges weakly in probab. to L_∞ with density

$$\frac{dL_\infty(x)}{dx} = \sum_{i \in \{-1,0,1\}} \sum_{1 \leqslant n \leqslant \infty} \mu^{(i)}(n) \frac{d\rho_n^{(i)}(x)}{dx},$$

where $\rho_n^{(i)}$, $i = 1, 0, -1$, is the expected spectral measure of $\sqrt{A_n A_n^*}$ where A_n is an $n \times n$ matrix from the real, complex and quaternionic Ginibre ensemble.
If X_g are Gaussian then a similar result holds for $L_{\mathcal{C}_N}$ (with $\theta_n^{(i)}$ in the limit).

Thank you