Maximum and decay of convolved densities

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Notations and setting

Given $X_1, \dots, X_n$ independent random vectors in $\mathbb{R}^d$, let

$$X = X_1 + \dots + X_n$$

(n does not need to tend to infinity).

Equivalent model: Weighted sums

$$Z_n = a_1 X_1 + \dots + a_n X_n, \qquad a_1^2 + \dots + a_n^2 = 1.$$

Problems: Assume that X_k have bounded densities p_k,

$$M_k = M(X_k) = \|p_k\|_\infty = \operatorname{ess\,sup}_x p_k(x),$$

so that X (or Z_n) has a bounded density p.

- How can one bound $M = M(X)$ in terms of M_k?
- What about $M(Z_n)$? (dependence on a_k)
- What is needed to control the decay of $p(x)$ at infinity?

(for example, moments of X_k / Laplace transform?)

One dimensional case

Theorem (Rogozin 1989). For fixed values $M_1, \dots, M_n$, $M = M(X)$ is maximized, when every $X_k \sim U(0, 1/M_k)$ is uniformly distributed on an interval of length $1/M_k$.

Theorem (Ball 1986). If $X_k \sim U(0,1)$, the maximum of density of $Z_n = a_1 X_1 + \dots + a_n X_n$ with $a_1^2 + \dots + a_n^2 = 1$ satisfies

$$1 \leq M(Z_n) \leq \sqrt{2}.$$

Geometric formulation (motivated by the Buseman-Petty problem 1956): For any hyperplane H in $\mathbb{R}^n$ passing through the center of $Q = [0,1]^n$,

$$1 \leq |Q \cap H| \leq \sqrt{2}.$$

Corollary. For $X = X_1 + \dots + X_n$,

$$\frac{1}{M^2} \geq \frac{1}{2} \sum_{k=1}^{n} \frac{1}{M_k^2}.$$

High dimensional case

Theorem (B-Chistyakov 2012). For $X = X_1 + \cdots + X_n$,

$$M(X)^{-\frac{2}{d}} \geq \frac{1}{e} \sum_{k=1}^{n} M(X_k)^{-\frac{2}{d}}.$$

Asymptotic equality: $X_k \sim U(B)$, $B = \{x \in \mathbb{R}^d : |x| < 1\}$, with $n, d \to \infty$.

Corollary. For $Z_n = a_1 X_1 + \cdots + a_n X_n$ with $M(X_k) \leq M$, $a_1^2 + \cdots + a_n^2 = 1$, we have

$$M(Z_n) \leq e^{d/2} M.$$

Rényi entropies

Let X be a random vector in $\mathbb{R}^d$ with density p

Rényi entropy of order $\alpha \in [0, \infty]$:

$$h_\alpha(X) = \frac{1}{1-\alpha} \log \int p^\alpha \, dx.$$

Rényi entropy power:

$$N_\alpha(X) = \exp\left\{\frac{2}{d} h_\alpha(X)\right\} = \left(\int p^\alpha \, dx\right)^{\frac{2}{d(1-\alpha)}}.$$

Limit cases:

$$h(X) = h_1(X) = -\int p \log p \, dx, \qquad N(X) = N_1(X) = \exp\left\{\frac{2}{d} h(X)\right\},$$

$$N_\infty(X) = M(X)^{-\frac{2}{d}}.$$

Monotonicity with respect to α:

$$N_\alpha(X) = \|p(X)\|_{\alpha-1}^{-2/d}.$$

Entropy power inequalities

Theorem (Shannon 1948, Stam 1959, Blachman 1965, Lieb 1978 ...) For $X = X_1 + \cdots + X_n$,

$$N(X) \geq \sum_{k=1}^{n} N(X_k).$$

Equality: Gaussian measures with proportional covariance matrices.
Theorem (B-Chistyakov 2015). For $X = X_1 + \cdots + X_n$ and $\alpha \geq 1$,

$$N_\alpha(X) \geq c_\alpha \sum_{k=1}^{n} N_\alpha(X_k).$$

One may take

$$c_\alpha = \alpha^{\frac{1}{\alpha-1}}.$$

Note: $\frac{1}{e} \leq c_\alpha \leq 1$, $c_\alpha \to 1$ as $\alpha \to 1$.

Further developments and refinements: Ram, Sason (2016), Li (2018), Melbourne, Tkocz (reversal inequalities 2021), Madiman, Melbourne, Roberto (discrete setting 2022)

Non-uniform bounds

As before, $X = X_1 + \cdots + X_n$ for independent r.v. in $\mathbb{R}^d$. Assume that X_k have bounded densities $p_k(x) \le M_k$.

Log-Laplace transform finite near zero:

$$V(t) = \log \mathbb{E}\, e^{\langle t, X\rangle}, \quad t \in \mathbb{R}^d$$

Legendre transform:

$$V^*(x) = \sup_{t \in \mathbb{R}^d} \, [\langle t, x\rangle - V(t)].$$

Theorem (B 2022). Assume that $\mathbb{E}X = 0$. If $n \ge 2$, the density p of X is continuous and satisfies, for all $x \in \mathbb{R}^d$,

$$p(x) \le M \exp\Big\{-\frac{1}{2}V^*(x)\Big\},$$

where M may be chosen as a function of $M_1, \dots, M_n$.

Choices of M

Geometric mean:

$$M = (M_1 \dots M_n)^{1/n}.$$

Based on harmonic mean:

Theorem. Suppose that for each $k \le n$,

$$\sum_{j \ne k} M_j^{-\frac{2}{d}} \ge M_k^{-\frac{2}{d}}.$$

Then one may take

$$M = e^{\frac{d}{2}} \left(M_1^{-\frac{2}{d}} + \dots + M_n^{-\frac{2}{d}} \right)^{-\frac{d}{2}}.$$

Application to maximum: Since

$$p(x) \le M \exp\left\{ -\frac{1}{2} V^*(x) \right\} \le M,$$

we get

$$M(X) \le e^{\frac{d}{2}} \left(M(X_1)^{-\frac{2}{d}} + \dots + M(X_n)^{-\frac{2}{d}} \right)^{-\frac{d}{2}}.$$

Proof

Let $d = 1$. Introduce

$$V_k(t) = \log \mathbb{E}\, e^{tX_k}, \quad k = 1, \dots, n,$$

so that the log-Laplace transform of $X = X_1 + \dots + X_n$ is given by

$$V(t) = V_1(t) + \dots + V_n(t).$$

Introduce the characteristic functions

$$f_k(t) = \mathbb{E}\, e^{itX_k} = \int e^{itx}\, p_k(x)\, dx, \quad t \in \mathbb{R},$$

$$f(t) = f_1(t) \dots f_n(t) = \int e^{itx}\, p(x)\, dx.$$

Using smoothing/multiplication by Gaussians, one may assume that $p(x)$ and $f(t)$ have a Gaussian decay at infinity. Then all V_k and V are everywhere finite, and $f(z)$ is an entire function, integrable along all axes $z = t - iy$ with fixed $y \in \mathbb{R}$.

Fourier inversion formula: For all

$$p(x) = \frac{1}{2\pi} \int e^{-itx} f(t)\, dt, \quad x \in \mathbb{R}.$$

Proof (cont.)

Contour integration: By Cauchy's theorem, for all $x, y \in \mathbb{R}$,

$$p(x) = \frac{e^{-yx}}{2\pi} \int e^{-itx} f(t - iy)\, dt.$$

Hence

$$p(x) \le \frac{e^{-yx}}{2\pi} \int |f(t - iy)|\, dt.$$

On this step, one can remove the assumption about the decay of p and f. Next, by Hölder's inequality,

$$\int |f(t - iy)|\, dt \le \prod_{k=1}^{n} \Big(\int |f_k(t - iy)|^{r_k}\, dt \Big)^{1/r_k}$$

for any collection $r_k > 1$ such that

$$\frac{1}{r_1} + \cdots + \frac{1}{r_n} = 1.$$

Here $t \to f_k(t - iy)$ represent Fourier transforms of

$$p_{k,y}(x) = e^{yx} p_k(x), \quad x \in \mathbb{R}.$$

Proof (cont.)

Hausdorff-Young inequality with optimal constants (Babenko 1961 for $r = 2, 4, 6, \ldots$ and Beckner 1975 in the general case): If $q \in L^{r'}$, $r \geq 2$, then its Fourier transform

$$\hat{q}(t) = \int e^{itx}\, q(x)\, dx, \quad t \in \mathbb{R},$$

belongs to L^r with norm

$$\|\hat{q}\|_r \leq (2\pi)^{1/r} A_r^{1/2}\, \|q\|_{r'}, \quad A_r = (r')^{1/r'}\, r^{-1/r}.$$

Choose $q = p_{k,y}$. If $r_k \geq 2$, then

$$\begin{aligned} \|f_k(t - iy)\|_{r_k} &\leq (2\pi)^{\frac{1}{r_k}} A_{r_k}^{1/2} \left(\int e^{r'_k yx} p_k(x)^{r'_k}\, dx \right)^{1/r'_k} \\ &\leq (2\pi)^{\frac{1}{r_k}} A_{r_k}^{1/2} \left(\int e^{r'_k yx} M_k^{r'_k - 1}\, p_k(x)\, dx \right)^{1/r'_k} \\ &= (2\pi)^{\frac{1}{r_k}} A_{r_k}^{1/2}\, M_k^{1/r_k} \exp\left\{ \frac{1}{r'_k} V_k(r'_k y) \right\}. \end{aligned}$$

Since $V_k \ge 0$ (due to $\mathbb{E} X_k = 0$) and $1 < r'_k \le 2$,

$$\frac{1}{r'_k}\, V_k(r'_k y) \le \frac{1}{2}\, V_k(2y).$$

Therefore

$$\int |f(t-iy)|\, dt \le 2\pi \prod_{k=1}^{n} A_{r_k}^{1/2}\, M_k^{1/r_k}\, \exp\Big\{\frac{1}{2}\, V(2y)\Big\}$$

and

$$\begin{aligned} p(x) \ &\le \ M \exp\Big\{ - yx + \frac{1}{2}\, V(2y)\Big\} \\ &= \ M \exp\Big\{ - [yx - \frac{1}{2}\, V(2y)]\Big\} \end{aligned}$$

with constant

$$M = \prod_{k=1}^{n} A_{r_k}^{1/2}\, M_k^{1/r_k}.$$

It remains to replace y with $y/2$ and optimize this inequality over all y's.

Choices of constants

Choosing $r_k = n$ and using $A_r \le 1$, we get

$$M = (M_1 \dots M_n)^{1/n}.$$

Alternatively, put $u_k = \frac{1}{r_k}$, $v_k = 1 - u_k$, and rewrite

$$A_{r_k} = \left(\frac{1}{r_k}\right)^{\frac{1}{r_k}} \left(\frac{1}{r'_k}\right)^{-\frac{1}{r'_k}} = u_k^{u_k} v_k^{-v_k}.$$

Then

$$M^2 = \prod_{k=1}^{n} u_k^{u_k} v_k^{-v_k} M_k^{2u_k}.$$

To simplify, one may take

$$u_k = \frac{M_k^{-2}}{M_1^{-2} + \dots + M_n^{-2}}$$

and then

$$M^{-2} = \left(M_1^{-2} + \dots + M_n^{-2}\right) \psi_n, \qquad \psi_n = \prod_{k=1}^{n} v_k^{v_k}.$$

Minimum to ψ_n on the simplex $u_1 + \dots + u_n = 1$, $u_k \ge 0$, is attained for $u_k = \frac{1}{n}$ with $\psi_n = (1 - \frac{1}{n})^{n-1} > \frac{1}{e}$. Note: $r_k \ge 2$, that is, $u_k \le \frac{1}{2}$ is equivalent to the condition of theorem.

Subgaussian distributions

A r.v. X in $\mathbb{R}^d$ is subgaussian, if

$$\mathbb{P}\{|X| \geq x\} \leq c_1 \, e^{-c_2 x^2}, \quad x \geq 0$$

(one may take $c_1 = 2$). Equivalently, when $\mathbb{E}X = 0$,

$$\mathbb{E}\, e^{\langle t, X \rangle} \leq e^{\sigma^2 |t|^2 / 2}, \quad t \in \mathbb{R}^d.$$

Dimension $d = 1$: Necessarily $\sigma^2 \geq \mathrm{Var}(X)$.

Possible particular case: $\mathbb{E}X = 0$, $\sigma^2 = \mathrm{Var}(X)$.

Sufficient condition: The characteristic function

$$f(z) = \mathbb{E}\, e^{iz\xi}, \quad z \in \mathbb{C},$$

is entire and has all roots ($f(z) = 0$) in $\mathrm{Re}(z) > 0$ in the angle $|\mathrm{Arg}(z)| \leq \frac{\pi}{8}$.

Examples: Convolution of Bernoulli and uniform distributions (all roots of f are real).

Subgaussian decay of convolved densities

Corollary. If X_k have subgaussian constants σ_k^2, then $X = X_1 + \cdots + X_n$ is subgaussian with

$$\sigma^2 = \sigma_1^2 + \cdots + \sigma_k^2.$$

Moreover, if $n \geq 2$ and $M(X_k) \leq M_k$, the density p of X satisfies

$$p(x) \leq M \exp\Big\{ -\frac{1}{4\sigma^2}\,|x|^2 \Big\}, \quad x \in \mathbb{R}^d.$$

Refinement. Given i.i.d. X_k in $\mathbb{R}^d$, let

$$Z_n = \frac{X_1 + \cdots + X_n}{\sqrt{n}}.$$

If $M(X_1) \leq M$ and $\sigma^2(X_1) \leq \sigma^2$, then the density q_n of Z_n satisfies

$$q_n(x) \leq e^{d/2} M \exp\Big\{ -\frac{n-1}{2n\sigma^2}\,|x|^2 \Big\}.$$

Rényi divergence

Relative α-entropy with respect to $Z \sim N(0, I_d)$ of order $\alpha > 0$:

$$D_\alpha(Z_n||Z) = \frac{1}{\alpha - 1} \log \int \left(\frac{q_n}{\varphi}\right)^\alpha \varphi \, dx.$$

The case $\alpha = 1$: Kullback-Leibler distance

$$D_1(Z_n||Z) = D(Z_n||Z) = \int q_n \log \frac{q_n}{\varphi} \, dx.$$

The case $\alpha = 2$: Equivalent to Pearson χ^2-distance.

Suppose that X_1 has mean zero and a unit covariance matrix, so that weakly $Z_n \Rightarrow Z$.

Entropic CLT (Barron 1986):

$$D(Z_n||Z) \to 0 \iff D(Z_n||Z) < \infty \quad \text{for some } n.$$

Proof based on de Bruijn's identity.
Fourier-analytic proof as a particular case of LLT in Orlicz spaces (B 2019).

CLT for Rényi divergence

Theorem (B-Chistyakov-Götze 2019). Given $1 < \alpha < \infty$, we have $D_\alpha(Z_n||Z) \to 0$, if and only if

$$\mathbb{E}\, e^{\langle t, X_1 \rangle} < e^{\alpha^* |t|^2/2}, \quad t \neq 0, \quad \alpha^* = \frac{\alpha}{\alpha - 1}$$

and

$$D_\alpha(Z_n||Z) < \infty \ \text{ for some } n.$$

Sufficient condition (using the subgaussian decay): If X_1 is subgaussian with $\sigma^2 = \sigma^2(X_1)$, we have $D_\alpha(Z_n||Z) \to 0$ for all

$$\alpha < \frac{\sigma^2}{\sigma^2 - 1}$$

(hence for all $\alpha > 1$ when $\sigma^2 = 1$).

Symmetric unimodal distributions

Corollary. Suppose that a r.v. X with an even density, which is non-increasing in $x > 0$. If $p(x) \le M$ and

$$\mathbb{P}\{|X| \ge x\} \le 2\, e^{-x^2/\sigma^2}, \quad x \ge 0,$$

then

$$p(x) \le C\sqrt{\frac{M}{\sigma}}\, e^{-cx^2/\sigma^2}.$$

Proof. The subgaussian r.v. $X' = X + \sigma U$, $U \sim U(-1,1)$, has density

$$q(x-\sigma) = \frac{1}{2\sigma}\int_{x-2\sigma}^{x} p(y)\, dy \ge p(x), \quad x \ge 2\sigma.$$

Example. Let a convex body $K \subset \mathbb{R}^n$ be isotropic and symmetric about coordinate axes, with volume one. Then, for $x \ge 0$,

$$\left| K \cap \{(x_1,\ldots,x_n) \in \mathbb{R}^n : x_1 + \cdots + x_n = x\sqrt{n}\,\}\right| \le Ce^{-cx^2}.$$

Bernoulli convolutions

Let $F_\lambda = \mathrm{Law}(Z_\lambda)$ of a random power series

$$Z_\lambda = \sqrt{1-\lambda^2} \sum_{k=0}^{\infty} \varepsilon_k \lambda^k, \quad 0 < \lambda < 1,$$

where $\varepsilon_k = \pm 1$ are independent Bernoulli. Normalization: $\mathbb{E} Z_\lambda^2 = 1$.

The case $0 < \lambda < \frac{1}{2}$: F_λ is singular
The case $\lambda = \frac{1}{2}$: F_λ is uniform

Problem: When does F_λ have density p_λ? If so, what can one say about its basic properties?

Erdós (1940): For any integer $m > 0$, there is $\lambda_m \in (\frac{1}{2}, 1)$ such that $p_\lambda \in C^m$ for almost all $\lambda \in (\lambda_m, 1)$.

Solomyak (1995): p_λ exists for almost all $\lambda \in (\frac{1}{2}, 1)$.

Shmerkin (2014): Exceptional λ's form a set $E \subset (\frac{1}{2}, 1)$ of Hausdorff dimension zero: $H_d(E) = 0$ for any $d > 0$.

Subgaussian decay of Bernoulli convolutions

Solomyak: $p_\lambda \in L^2$ for almost all $\lambda \in (\frac{1}{2}, 1)$. Equivalently, the characteristic function

$$f_\lambda(t) = \mathbb{E}\, e^{itZ_\lambda} = \prod_{k=0}^{\infty} \cos\left(\sqrt{1-\lambda^2}\,\lambda^k t\right)$$

is square integrable, that is,

$$I(\lambda) = \int_{-\infty}^{\infty} f_\lambda(t)^2\, dt < \infty.$$

Proposition. For almost all $\lambda \in (2^{-1/4}, 1)$, the density p_λ is continuous and admits the subgaussian upper bound

$$p_\lambda(x) < \frac{1}{4}\, I(\lambda^4)\, e^{-x^2/4}, \quad x \in \mathbb{R}.$$

Note: By the Berry-Esseen theorem,

$$\sup_x |F_\lambda(x) - \Phi(x)| \le c\sqrt{1-\lambda^2}$$

for all $\lambda \in (0, 1)$.