Strong asymptotic freeness for independent uniform variables on compact groups

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Representations in high dimension

Unitary representation

Let G be a compact group and ρ be a unitary representation of G in $\mathbb{C}^N$, that is, for $g, h \in G$,

$$\rho(g) \in \mathbb{U}_N\,, \quad \rho(e) = I_N \quad \text{and} \quad \rho(g \cdot h) = \rho(g) \cdot \rho(h).$$

For example, if $G = \mathbb{U}_n$, $\mathbb{O}_n$ or $\mathbb{S}_n$:

- ⋆ standard rep : $N = n$, $\rho(U) = U$,
- ⋆ determinant : $N = 1$, $\rho(U) = \det(U)$,
- ⋆ tensor product : $N = n^{q_+ + q_-}$, $\rho(U) = U^{\otimes q_+} \otimes \bar{U}^{\otimes q_-}$.

Representations in high dimension

We consider a sequence of representations ρ_N of growing dimensions.

Non-commutative probability offers a natural framework to describe limits of representations of high dimensions, *Biane (1995,1998).*

Here, we explore a probabilistic direction and study the representations by sampling group elements from the Haar measure.

Non-commutative probability space

NCPS : pair formed by a unital algebra $\mathcal{A}$ and a linear functional $\tau : \mathcal{A} \to \mathbb{C}$ such that $\tau(1) = 1$. E.g. $\mathcal{A} = M_N(\mathbb{C})$, $\tau = \frac{1}{N}\mathrm{Tr}$.

$\mathcal{A}$ is a $\star$-algebra : it comes with a linear involution such that $(ab)^* = b^* a^*$ and $\tau(a^*) = \overline{\tau(a)}$.

Assume that $\tau(aa^*) \geqslant 0$ with equality iff $a = 0$ (positive and faithful).

We may define the norms

$$\|a\|_2 = \sqrt{\tau(aa^*)} \quad \text{and} \quad \|a\| = \sup\{\|ab\|_2 : \|b\|_2 \leqslant 1\}.$$

Non-commutative probability space

In this talk, only two examples of NCPS.

Matrices : $\mathcal{A} = M_N(\mathbb{C})$, $\tau = \frac{1}{N}\mathrm{Tr}$ and $\|\cdot\|$ is the operator norm.

Group algebra : G a countable group and $\mathcal{A}$ the algebra on $\ell^2(G)$ generated by the left multiplication operators :

$$\lambda(g)(\delta_h) = \delta_{gh}.$$

That is, λ is the left regular representation. It is unitary.

For $\tau = \langle \delta_e, \cdot\, \delta_e \rangle$, $\|\cdot\|$ coincides with the operator norm on $\ell^2(G)$.

Convergence in NC probability spaces

Let $(v_{1,N}, \ldots, v_{d,N})$ be elements of a NCPS $(\mathcal{A}_N, \tau_N)$ and $(v_1, \ldots, v_d)$ be elements of a NCPS $(\mathcal{A}, \tau)$.

Convergence in NC distribution : for any NC polynomial P in d variables and their adjoints,

$$\tau_N\left(P(v_{1,N}, \ldots, v_{d,N})\right) \to \tau\left(P(v_1, \ldots, v_d)\right).$$

E.g. $P = v_1 v_2 - v_2 v_1$, $P = v_1 + v_1^* + v_2 + v_2^*$ or $P = (v_1 + v_2)(v_1 + v_2)^*$.

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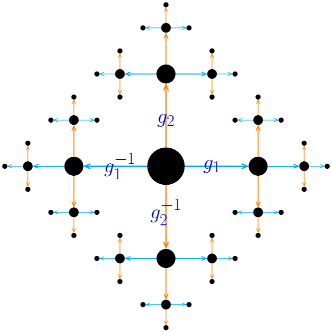
$$\|P(v_{1,N}, \ldots, v_{d,N})\| \to \|P(v_1, \ldots, v_d)\|.$$

If $(v_1, \ldots, v_d)$ are free one speaks of (strong) asymptotic freeness.

Free unitaries

Let $\mathbb{F}_d$ be the free group with d free generators $g_1, \ldots, g_d$.



The unitary operators on $\ell^2(\mathbb{F}_d)$, $(\lambda(g_1), \ldots, \lambda(g_d))$ are free.

Voiculescu's freeness

Let $(\mathcal{A}, \tau)$ be a NCPS.

Sub-algebras $(\mathcal{A}_1, \ldots, \mathcal{A}_d)$ are free if

$$\tau(a_1 a_2 \cdots a_k) = 0,$$

whenever $\tau(a_i) = 0$ and $a_i \in \mathcal{A}_{l_i}$, $l_i \neq l_{i+1}$.

Elements $(v_1, \ldots, v_d)$ are free if the subalgebras they span are free.

Convergence in NC probability spaces

Asymptotic freeness :

- iid GUE matrices, *Voiculescu (1991)*,

- iid Haar distributed unitary matrices, *Voiculescu (1998)*,

- iid uniform permutation matrices, *Nica (1993)* ,

- ...

Strong asymptotic freeness :

- iid GUE matrices, *Haagerup-Thorbjørnsen (2005)*, *Schultz, Capitaine-Donati, Male, Anderson, Collins-Guionnet-Parraud, Bandeira-Boedihardjo-van Handel, ...*

- iid Haar distributed unitary matrices, *Collins-Male (2012)*,

- iid uniform permutation matrices, *B-Collins (2019)*,

- tensor product iid GUE matrices, *Belinschi-Capitaine (2022)*.

The quest of strong convergence

Strong convergence has always very powerful consequences :

- ⋆ $\mathrm{Ext}(C^*_{\mathrm{red}}(F_2))$ is not a group *Haagerup-Thorbjørnsen (2005).*
- ⋆ The generalized Alon's conjecture *B-Collins (2019).*
- ⋆ Cut-off for finite space Markov chains *B-Lacoin (2020).*
- ⋆ Hayes' approach to Peterson-Thom conjecture *Belinschi-Capitaine (2022).*

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BUT : no known non-trivial deterministic examples and few random examples.

Representations in high dimension

Back to our setting : G a compact group and ρ_N a unitary rep on $\mathbb{C}^N$.

We consider $U_1, \ldots, U_d$ be iid uniform sampled according to the Haar measure on G and set

$$V_{i,N} = \rho_N(U_i) \in \mathbb{U}_N.$$

Question : along a sequence $N \to \infty$, convergence of $(V_{1,N}, \ldots, V_{d,N})$?

In this talk : $G = \mathbb{U}_n$, $\mathbb{O}_n$ or $\mathbb{S}_n$ and (n, N) both go to infinity.

Alternative motivation I :
Random representations of the free group

Consider the free group $\mathbb{F}_d$ with d free generators $g_1, \ldots, g_d$.

We set

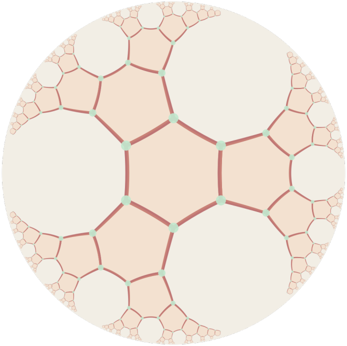
$$\rho_N^{\text{free}}(g_i) = V_{i,N} = \rho_N(U_i) \in \mathbb{U}_N.$$

It extends uniquely to a unitary representation of $\mathbb{F}_d$ on $\Gamma_N = \rho_N(G)$: $\rho_N^{\text{free}}(g_1 g_2 g_1^{-1}) = V_1 V_2 V_1{}^*$. If Γ_N is a subgroup of $\mathbb{S}_N$ one speaks of an action of $\mathbb{F}_d$ on $\{1, \ldots, N\}$.

This random representation is uniform. How close is it to the regular representation of $\mathbb{F}_d$?

Alternative motivation II : Optimal expander graphs

Construct strong finite dim approximations of operators like this one :



Courtesy of Ryan O'Donnell and Xinyu Wu

Alternative motivation III : Quantum expanders

For $G = \mathbb{U}_n$ or $\mathbb{O}_n$, in quantum info theory, the norm of operators like

$$\sum_{i=1}^{d} U_i \otimes \bar{U}_i + (U_i \otimes \bar{U}_i)^*$$

appears *Hastings, Harrow, Pisier*.

Strong asymptotic freeness for representations of the unitary group

Tensor product representation

Set $G = \mathbb{U}_n$ and consider the rep on $\mathbb{C}^N$, with $N = n^q$, $q_+ + q_- = q$,

$$V = U^{\otimes q_+} \otimes \bar{U}^{\otimes q_-}.$$

If $q_- = q_+$, it has fixed points for all $U \in \mathbb{U}_n$. E.g. for $\mathbb{C}^{n^2} \simeq M_n(\mathbb{C})$,

$$(U \otimes \bar{U}) I_n = U I_n U^* = I_n.$$

We denote by H the vector subspace of $\mathbb{C}^N$ of such fixed points (fully explicit of $\dim(H) = q_+!$).

Tensor product representation

Let $(U_1, \dots, U_d)$ be iid Haar distributed on $\mathbb{U}_n$ and

$$V_i = U_i^{\otimes q_+} \otimes \bar{U}_i^{\otimes q_-}.$$

Theorem
Restricted to $H^\perp$, a.s. $(V_1, \dots, V_d)$ are strongly asymptotically free as $n \to \infty$ and $q = q_+ + q_- \leqslant c \ln(n) / \ln\ln(n)$.

The same statement holds for $\mathbb{O}_n$. Also, for $\mathbb{S}_n$ and $q = 1, 2$.

Tensor product representation

Asymptotic freeness follows from *Voiculescu (1998)*, *Collins-Gaudreau Lamarre-Male (2017, 2020)*.

The only obstructions to strong freeness are the fixed points in H.

The simplest case $P = \sum_i V_i + V_i^*$ treated in *Harrow-Hastings (2009)*.

Related deterministic result in *Bourgain-Gamburd (2012)*.

Irreducible representation

A rep of G is irreducible if it has no non-trivial stable subspace.

Irreducible rep ρ of $\mathbb{U}_n$ are indexed by a signature : a pair of Young diagrams (λ, μ), two non-increasing sequences of integers $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0$ with length $|\rho| = \sum_i \lambda_i + \mu_i \leqslant n$.

$U \leftrightarrow ((1), 0)$, $\bar{U} \leftrightarrow (0, (1))$, $\det(U) \leftrightarrow ((1, \dots, 1), 0)$, ...

Irreducible representation

Let $(U_1, \ldots, U_d)$ be iid Haar distributed on $\mathbb{U}_n$ and ρ an irreducible representation. We set

$$V_i = \rho(U_i).$$

Corollary

A.s. $(V_1, \ldots, V_d)$ *are strongly asymptotically free as* $n \to \infty$ *and* $1 \leqslant |\rho| \leqslant c \ln(n) / \ln \ln(n)$.

Indeed, ρ is a sub-representation of a tensor rep with $q = |\rho|$.

Result cannot hold for all rep : $\det(U)$ is one-dim and commutative.

Outline of proof

Analysis vs combinatorics

Strong asymptotic freeness results all relied on analysis : linearization, analytic properties of resolvent (Schwinger-Dyson - loop equation), complex analysis, interpolation methods.

In *B-Collins (2019)* results for random permutations rely on moments through new techniques.

For tensor products of Haar unitaries, we expand these techniques.

Strategy

1. *Centering.*
2. *Linearization trick.*
3. *Matrix-valued nonbacktracking operators.*
4. *Fűredi-Komlós expected high trace method.*
5. *High order centered Weingarten calculus.*

Centering

$(U_1, \ldots, U_d)$ iid Haar distributed on $\mathbb{U}_n$ and

$$V_i = U_i^{\otimes q_+} \otimes \bar{U}_i^{\otimes q_-}.$$

H = vector subspace of $\mathbb{C}^N$ of fixed points of this tensor rep.

We want to prove the strong asymp freeness of $(V_1, \ldots, V_d)$ on $H^\perp$.

We have

$$\mathbb{E}V_i = \mathrm{Proj}_H.$$

We need to prove the strong asymp freeness of $([V_1], \ldots, [V_d])$ with

$$[V_i] = V_i - \mathbb{E}V_i.$$

Linearization trick

Let $(v_1, \ldots, v_d)$ in a NCPS $(\mathcal{A}, \tau)$ and $(V_1, \ldots, V_d)$ in $M_N(\mathbb{C})$. For $i = 1, \ldots, d$, set $i^* = i + d$, $i^{**} = i$,

$$V_{i+d} = V_{i^*} = V_i^*, \quad v_{i+d} = v_{i^*} = v_i^*.$$

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$$V_{i+d} = V_{i^*} = V_i^*, \quad v_{i+d} = v_{i^*} = v_i^*.$$

The convergence for all NC polynomials,

$$\|P(V_1, \ldots, V_{2d})\| \to \|P(v_1, \ldots, v_{2d})\|$$

is equivalent to : for all integers $r \geqslant 1$ and $a_i \in M_r(\mathbb{C})$, $a_{i^*} = a_i^*$,

$$\left\| a_0 \otimes I + \sum_{i=1}^{2d} a_i \otimes V_i \right\| \to \left\| a_0 \otimes 1 + \sum_{i=1}^{2d} a_i \otimes v_i \right\|.$$

It suffices to consider matrix-valued polynomials of degree one!

Pisier (1996), Haagerup-Thorbjørnsen (2005)

Matrix-valued nonbacktracking operators

Assume now that $(v_1, \ldots, v_d)$ are free unitaries and $V_i \in \mathbb{U}_N$, $a_i \in M_r(\mathbb{C})$,

$$A = a_0 \otimes I + \sum_{i=1}^{2d} a_i \otimes V_i.$$

For $b_i \in M_r(\mathbb{C})$, set $E_{ij} = e_i \otimes e_j \in M_{2d}(\mathbb{C})$ and

$$B = \sum_{(i,j):i \neq j^*} b_i \otimes V_i \otimes E_{ij}.$$

The convergence of the spectral radii of all nonbactracking operators implies the convergence of the spectrum of A*.*

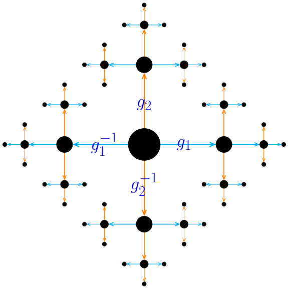
Nonbacktracking operators

$$A = a_0 \otimes I + \sum_i a_i \otimes V_i \quad \text{vs} \quad B = \sum_{(i,j):i \neq j^*} b_i \otimes V_i \otimes E_{ij}.$$

On $\mathbb{F}_d$, powers of B follows geodesics : for $V_i = v_i = \lambda(g_i)$,

$$B^k \phi \otimes \delta_e \otimes \delta_j = \sum_{g=(g_{i_1},\ldots,g_{i_k})} \phi_g \otimes \delta_g \otimes \delta_{i_k},$$

with $(g_{i_1}, \ldots, g_{i_k})$ reduced, $i_l \neq i^*_{l-1}, i_0 = j$.



Expected high trace method

$$B = \sum_{(i,j):i \neq j^*} b_i \otimes [V_i] \otimes E_{ij} \ , \quad B_{\text{free}} = \sum_{(i,j):i \neq j^*} b_i \otimes v_i \otimes E_{ij} .$$

Goal : compare the spectral radii of B and B_{free} for all values of (b_i).

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Goal : compare the spectral radii of B and B_{free} for all values of (b_i).

For fixed (b_i) and $k \gg \ln(N)$,

$$\mathbb{E}\rho(B)^{2k} \leqslant \mathbb{E}\|B^k B^{k*}\| \leqslant \mathbb{E}\mathrm{Tr}(B^k B^{k*}) \overset{?}{\leqslant} N(1+o(1))^k \rho(B_{\text{free}})^{2k}.$$

We expand the trace as a sum of weighted paths, two ingredients :

- combinatoric of paths with the tensor structure of the V_i's,
- average of product of $2k$ entries of the V_i's.

A final net argument to have a joint probabilistic estimate for all (b_i).

Centered Weingarten Calculus

For a random variable X, define $[X] = X - \mathbb{E}X$.

Let $U \in \mathbb{O}_n$ Haar distributed. We want to compute an expression like

$$\mathbb{E}\prod_{t=1}^{T}\left[\prod_{p=1}^{q} U_{i_{t,p}j_{t,p}}\right]$$

in a meaningful way with $k = qT$ large.

Centered Weingarten Calculus

Wick calculus for Gaussian moments has an analog for unitary groups.

We can write a Weingarten formula

$$\mathbb{E}\prod_{t=1}^{T}\left[\prod_{p=1}^{q}U_{i_{tp}j_{tp}}\right]=\sum_{\sigma,\tau}\delta_\sigma(i)\delta_\tau(j)[\mathrm{Wg}](\sigma,\tau),$$

where the sum is over all pairs (σ,τ) of pairings of

$$I=\{(t,p):1\leqslant t\leqslant T,1\leqslant p\leqslant q\},$$

and $\delta_\sigma(i)$ is 1 if σ matches the same indices of i and 0 otherwise.

The expression of $[\mathrm{Wg}](\sigma,\tau)$ is complicated but expanding on *Collins-Matsumoto (2017)* we have upper and lower bounds.

Haar unitary vs Gaussian

Let G_{ij} be iid standard Gaussian variables.

Theorem

For $k = qT \leqslant n^{4/7}$,

$$n^{k/2}\left|\mathbb{E}\prod_{t=1}^{T}\left[\prod_{p=1}^{q}U_{i_{tp}j_{tp}}\right]\right| \leqslant (1+\delta)\mathbb{E}\prod_{t=1}^{T}\left(\left[\prod_{p=1}^{q}G_{i_{tp}j_{tp}}\right]+\eta\right),$$

with $\delta = 3k^{7/2}n^{-2}$ *and* $\eta = 2k^{q}n^{-1/2}$.

The right-hand side can be estimated by using Wick calculus.

Concluding words

Further directions

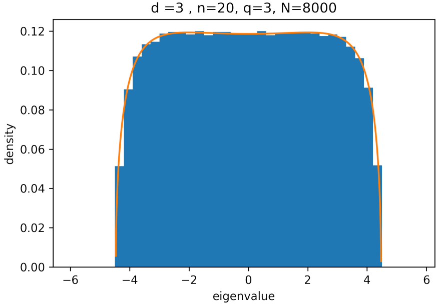
Among open directions :

- ⋆ non-asymptotic bounds *(Bandeira-Boedihardjo-van Handel).*
- ⋆ tensor product of permutation matrices,
- ⋆ random and deterministic unitary matrices,
- ⋆ replace the free group by other non-amenable groups, such as surface groups *(Magee, Naud, Puder)* or free products of finite groups *(Puder, Zimhoni).*
- ⋆ what happens for n fixed and $q \to \infty$?

Simulation

For $(U_1, \ldots, U_d)$ iid Haar on $\mathbb{SU}_n$ and

$$\sum_{i=1}^{d} U_i^{\otimes q} + U_i^{*\otimes q}.$$



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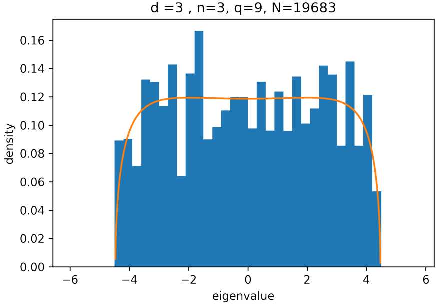

Thank you for your attention !