Strong asymptotic freeness for independent uniform variables on compact groups

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Joint work with Benoît Collins (Kyoto University)
Representations in high dimension
Unitary representation

Let $G$ be a compact group and $\rho$ be a unitary representation of $G$ in $\mathbb{C}^N$, that is, for $g, h \in G$,

$$\rho(g) \in \mathbb{U}_N, \quad \rho(e) = I_N \quad \text{and} \quad \rho(g \cdot h) = \rho(g) \cdot \rho(h).$$

For example, if $G = \mathbb{U}_n, \mathbb{O}_n$ or $\mathbb{S}_n$ :

- standard rep : $N = n$, $\rho(U) = U$,
- determinant : $N = 1$, $\rho(U) = \det(U)$,
- tensor product : $N = n^{q+}q^-$, $\rho(U) = U^{\otimes q+} \otimes \bar{U}^{\otimes q-}$.
Representations in high dimension

We consider a sequence of representations $\rho_N$ of growing dimensions.


Here, we explore a probabilistic direction and study the representations by sampling group elements from the Haar measure.
Non-commutative probability space

NCPS : pair formed by a unital algebra $\mathcal{A}$ and a linear functional $\tau : \mathcal{A} \rightarrow \mathbb{C}$ such that $\tau(1) = 1$. E.g. $\mathcal{A} = M_N(\mathbb{C})$, $\tau = \frac{1}{N} \text{Tr}$.

$\mathcal{A}$ is a $\star$-algebra : it comes with a linear involution such that $(ab)^\ast = b^\ast a^\ast$ and $\tau(a^\ast) = \overline{\tau(a)}$.

Assume that $\tau(aa^\ast) \geq 0$ with equality iff $a = 0$ (positive and faithful).

We may define the norms

$$\|a\|_2 = \sqrt{\tau(aa^\ast)} \quad \text{and} \quad \|a\| = \sup\{\|ab\|_2 : \|b\|_2 \leq 1\}.$$
Non-commutative probability space

In this talk, only two examples of NCPS.

**Matrices** : \( \mathcal{A} = M_N(\mathbb{C}) \), \( \tau = \frac{1}{N} \text{Tr} \) and \( \| \cdot \| \) is the operator norm.

**Group algebra** : \( G \) a countable group and \( \mathcal{A} \) the algebra on \( \ell^2(G) \) generated by the left multiplication operators :

\[
\lambda(g)(\delta_h) = \delta_{gh}.
\]

That is, \( \lambda \) is the left regular representation. It is unitary.

For \( \tau = \langle \delta_e, \cdot \delta_e \rangle \), \( \| \cdot \| \) coincides with the operator norm on \( \ell^2(G) \).
Convergence in NC probability spaces

Let \((v_{1,N}, \ldots, v_{d,N})\) be elements of a NCPS \((\mathcal{A}_N, \tau_N)\) and \((v_1, \ldots, v_d)\) be elements of a NCPS \((\mathcal{A}, \tau)\).

**Convergence in NC distribution**: for any NC polynomial \(P\) in \(d\) variables and their adjoints,

\[
\tau_N (P(v_{1,N}, \ldots, v_{d,N})) \to \tau(P(v_1, \ldots, v_d)).
\]

E.g. \(P = v_1 v_2 - v_2 v_1\), \(P = v_1 + v_1^* + v_2 + v_2^*\) or \(P = (v_1 + v_2)(v_1 + v_2)^*\).
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Strong convergence: in addition,

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\|P(v_{1,N}, \ldots, v_{d,N})\| \to \|P(v_1, \ldots, v_d)\|.
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If \((v_1, \ldots, v_d)\) are free one speaks of (strong) asymptotic freeness.
Free unitaries

Let $\mathbb{F}_d$ be the free group with $d$ free generators $g_1, \ldots, g_d$.

The unitary operators on $\ell^2(\mathbb{F}_d), (\lambda(g_1), \ldots, \lambda(g_d))$ are free.
Let \((\mathcal{A}, \tau)\) be a NCPS.

Sub-algebras \((\mathcal{A}_1, \ldots, \mathcal{A}_d)\) are free if

\[\tau(a_1a_2 \cdots a_k) = 0,\]

whenever \(\tau(a_i) = 0\) and \(a_i \in \mathcal{A}_{l_i}, l_i \neq l_{i+1}\).

Elements \((v_1, \ldots, v_d)\) are free if the subalgebras they span are free.
Convergence in NC probability spaces

Asymptotic freeness:

- iid GUE matrices, \textit{Voiculescu (1991)},
- iid Haar distributed unitary matrices, \textit{Voiculescu (1998)},
- iid uniform permutation matrices, \textit{Nica (1993)},
- ...

Strong asymptotic freeness:

- iid GUE matrices, \textit{Haagerup-Thorbjørnsen (2005)}, Schultz, Capitaine-Donati, Male, Anderson, Collins-Guionnet-Parraud, Bandeira-Boedihardjo-van Handel, ...
- iid Haar distributed unitary matrices, \textit{Collins-Male (2012)},
- iid uniform permutation matrices, \textit{B-Collins (2019)},
- tensor product iid GUE matrices, \textit{Belinschi-Capitaine (2022)}.
The quest of strong convergence

Strong convergence has always very powerful consequences:

⋆ $\text{Ext}(C^*_\text{red}(F_2))$ is not a group *Haagerup-Thorbjørnsen (2005).*

⋆ The generalized Alon’s conjecture *B-Collins (2019).*

⋆ Cut-off for finite space Markov chains *B-Lacoin (2020).*

⋆ Hayes’ approach to Peterson-Thom conjecture *Belinschi-Capitaine (2022).*
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BUT: no known non-trivial deterministic examples and few random examples.
Representations in high dimension

Back to our setting: \( G \) a compact group and \( \rho_N \) a unitary rep on \( \mathbb{C}^N \).

We consider \( U_1, \ldots, U_d \) be iid uniform sampled according to the Haar measure on \( G \) and set

\[
V_{i,N} = \rho_N(U_i) \in \mathbb{U}_N.
\]

Question: along a sequence \( N \to \infty \), convergence of \( (V_{1,N}, \ldots, V_{d,N}) \)?

In this talk: \( G = \mathbb{U}_n, \mathbb{O}_n \) or \( \mathbb{S}_n \) and \( (n, N) \) both go to infinity.
Alternative motivation I:
Random representations of the free group

Consider the free group $\mathbb{F}_d$ with $d$ free generators $g_1, \ldots, g_d$.

We set

$$\rho_N^\text{free}(g_i) = V_{i,N} = \rho_N(U_i) \in \mathbb{U}_N.$$ 

It extends uniquely to a unitary representation of $\mathbb{F}_d$ on $\Gamma_N = \rho_N(G) : \rho_N^\text{free}(g_1g_2g_1^{-1}) = V_1V_2V_1^*$. If $\Gamma_N$ is a subgroup of $\mathbb{S}_N$ one speaks of an action of $\mathbb{F}_d$ on $\{1, \ldots, N\}$.

This random representation is uniform. How close is it to the regular representation of $\mathbb{F}_d$?
Alternative motivation II:
Optimal expander graphs

Construct strong finite dim approximations of operators like this one:

Courtesy of Ryan O’Donnell and Xinyu Wu
Alternative motivation III: Quantum expanders

For $G = \mathbb{U}_n$ or $\mathbb{O}_n$, in quantum info theory, the norm of operators like

$$\sum_{i=1}^{d} U_i \otimes \bar{U}_i + (U_i \otimes \bar{U}_i)^{*}$$

appears Hastings, Harrow, Pisier.
Strong asymptotic freeness for representations of the unitary group
Tensor product representation

Set $G = \mathbb{U}_n$ and consider the rep on $\mathbb{C}^N$, with $N = n^q$, $q_+ + q_- = q$,

$$V = U^{\otimes q_+} \otimes \bar{U}^{\otimes q_-}.$$ 

If $q_- = q_+$, it has fixed points for all $U \in \mathbb{U}_n$. E.g. for $\mathbb{C}^{n^2} \simeq M_n(\mathbb{C})$,

$$(U \otimes \bar{U})I_n = UI_nU^* = I_n.$$ 

We denote by $H$ the vector subspace of $\mathbb{C}^N$ of such fixed points (fully explicit of $\text{dim}(H) = q_+$).
Let \((U_1, \ldots, U_d)\) be iid Haar distributed on \(\mathbb{U}_n\) and

\[
V_i = U_i^{\otimes q_+} \otimes \bar{U}_i^{\otimes q_-}.
\]

**Theorem**

Restricted to \(H^\perp\), a.s. \((V_1, \ldots, V_d)\) are strongly asymptotically free as \(n \to \infty\) and \(q = q_+ + q_- \leq c \ln(n) / \ln \ln(n)\).

The same statement holds for \(\mathbb{O}_n\). Also, for \(\mathbb{S}_n\) and \(q = 1, 2\).
Tensor product representation


The only obstructions to strong freeness are the fixed points in $H$.

The simplest case $P = \sum_i V_i + V_i^*$ treated in Harrow-Hastings (2009).

Related deterministic result in Bourgain-Gamburd (2012).
Irreducible representation

A rep of $G$ is **irreducible** if it has no non-trivial stable subspace.

Irreducible rep $\rho$ of $\mathbb{U}_n$ are indexed by a signature: a pair of Young diagrams $(\lambda, \mu)$, two non-increasing sequences of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ with length $|\rho| = \sum_i \lambda_i + \mu_i \leq n$.

$U \leftrightarrow ((1), 0), \bar{U} \leftrightarrow (0, (1)), \det(U) \leftrightarrow ((1, \ldots, 1), 0), \ldots$
Irreducible representation

Let \((U_1, \ldots, U_d)\) be iid Haar distributed on \(\mathbb{U}_n\) and \(\rho\) an irreducible representation. We set

\[ V_i = \rho(U_i). \]

**Corollary**

A.s. \((V_1, \ldots, V_d)\) are strongly asymptotically free as \(n \to \infty\) and

\[ 1 \leq |\rho| \leq c \ln(n) / \ln \ln(n). \]

Indeed, \(\rho\) is a sub-representation of a tensor rep with \(q = |\rho|\).

Result cannot hold for all rep : \(\det(U)\) is one-dim and commutative.
Outline of proof
Analysis vs combinatorics

Strong asymptotic freeness results all relied on analysis: linearization, analytic properties of resolvent (Schwinger-Dyson - loop equation), complex analysis, interpolation methods.

In *B-Collins (2019)* results for random permutations rely on moments through new techniques.

For tensor products of Haar unitaries, we expand these techniques.
Strategy

1. Centering.

2. Linearization trick.


4. Fűredi-Komlós expected high trace method.

5. High order centered Weingarten calculus.
Centering

$(U_1, \ldots, U_d)$ iid Haar distributed on $\mathbb{U}_n$ and

$$V_i = U_i^\otimes q^+ \otimes \bar{U}_i^\otimes q^-.$$  

$H = $ vector subspace of $\mathbb{C}^N$ of fixed points of this tensor rep.

We want to prove the strong asymp freeness of $(V_1, \ldots, V_d)$ on $H^\perp$.

We have

$$\mathbb{E}V_i = \text{Proj}_H.$$ 

We need to prove the strong asymp freeness of $([V_1], \ldots, [V_d])$ with

$$[V_i] = V_i - \mathbb{E}V_i.$$
Linearization trick

Let $(v_1, \ldots, v_d)$ in a NCPS $(\mathcal{A}, \tau)$ and $(V_1, \ldots, V_d)$ in $M_N(\mathbb{C})$. For $i = 1, \ldots, d$, set $i^* = i + d$, $i^{**} = i$,

$$V_{i+d} = V_{i^*} = V_i^*, \quad v_{i+d} = v_{i^*} = v_i^*.$$
Linearization trick

Let \((v_1, \ldots, v_d)\) in a NCPS \((\mathcal{A}, \tau)\) and \((V_1, \ldots, V_d)\) in \(M_N(\mathbb{C})\). For \(i = 1, \ldots, d\), set \(i^* = i + d\), \(i^{**} = i\),

\[
V_{i+d} = V_{i^*} = V_{i}^*, \quad v_{i+d} = v_{i^*} = v_{i}^*.
\]

The convergence for all NC polynomials,

\[
\|P(V_1, \ldots, V_{2d})\| \to \|P(v_1, \ldots, v_{2d})\|
\]

is equivalent to: for all integers \(r \geq 1\) and \(a_i \in M_r(\mathbb{C})\), \(a_{i^*} = a_{i}^*\),

\[
\left\| a_0 \otimes I + \sum_{i=1}^{2d} a_i \otimes V_i \right\| \to \left\| a_0 \otimes 1 + \sum_{i=1}^{2d} a_i \otimes v_i \right\|.
\]

It suffices to consider matrix-valued polynomials of degree one!

*Pisier (1996), Haagerup-Thorbjørnsen (2005)*
Matrix-valued nonbacktracking operators

Assume now that \((v_1, \ldots, v_d)\) are free unitaries and \(V_i \in U_N\), \(a_i \in M_r(\mathbb{C})\),

\[
A = a_0 \otimes I + \sum_{i=1}^{2d} a_i \otimes V_i.
\]

For \(b_i \in M_r(\mathbb{C})\), set \(E_{ij} = e_i \otimes e_j \in M_{2d}(\mathbb{C})\) and

\[
B = \sum_{(i,j): i \neq j^*} b_i \otimes V_i \otimes E_{ij}.
\]

The convergence of the spectral radii of all nonbactracking operators implies the convergence of the spectrum of \(A\).
Nonbacktracking operators

\[ A = a_0 \otimes I + \sum_i a_i \otimes V_i \quad \text{vs} \quad B = \sum_{(i,j):i \neq j^*} b_i \otimes V_i \otimes E_{ij}. \]

On \( \mathbb{F}_d \), powers of \( B \) follows geodesics: for \( V_i = v_i = \lambda(g_i) \),

\[ B^k \phi \otimes \delta_e \otimes \delta_j = \sum_{g=(g_{i_1}, \ldots, g_{i_k})} \phi_g \otimes \delta_g \otimes \delta_{i_k}, \]

with \((g_{i_1}, \ldots, g_{i_k})\) reduced, \( i_l \neq i_{l-1}, i_0 = j \).
Expected high trace method

\[ B = \sum_{(i,j): i \neq j^*} b_i \otimes [V_i] \otimes E_{ij}, \quad B_{\text{free}} = \sum_{(i,j): i \neq j^*} b_i \otimes v_i \otimes E_{ij}. \]

Goal: compare the spectral radii of \( B \) and \( B_{\text{free}} \) for all values of \( (b_i) \).
Expected high trace method

\[ B = \sum_{(i,j): i\neq j^*} b_i \otimes [V_i] \otimes E_{ij}, \quad B_{\text{free}} = \sum_{(i,j): i\neq j^*} b_i \otimes v_i \otimes E_{ij}. \]

Goal: compare the spectral radii of \( B \) and \( B_{\text{free}} \) for all values of \((b_i)\).

For fixed \((b_i)\) and \(k \gg \ln(N)\),

\[ \mathbb{E} \rho(B)^{2k} \leq \mathbb{E} \| B^k B^{k\ast} \| \leq \mathbb{E} \text{Tr}(B^k B^{k\ast}) \leq N(1 + o(1))^k \rho(B_{\text{free}})^{2k}. \]

We expand the trace as a sum of weighted paths, two ingredients:

- combinatoric of paths with the tensor structure of the \( V_i \)'s,
- average of product of \( 2k \) entries of the \( V_i \)'s.

A final net argument to have a joint probabilistic estimate for all \((b_i)\).
For a random variable $X$, define $[X] = X - \mathbb{E}X$.

Let $U \in \mathcal{O}_n$ Haar distributed. We want to compute an expression like

$$\mathbb{E} \prod_{t=1}^{T} \left[ \prod_{p=1}^{q} U_{i_t,p} j_{t,p} \right]$$

in a meaningful way with $k = qT$ large.
Weingarten Calculus

**Wick calculus** for Gaussian moments has an analog for unitary groups.

We can write a Weingarten formula

\[
\mathbb{E} \prod_{t=1}^{T} \left[ \prod_{p=1}^{q} U_{i_{tp},j_{tp}} \right] = \sum_{\sigma,\tau} \delta_{\sigma}(i) \delta_{\tau}(j) [Wg](\sigma, \tau),
\]

where the sum is over all pairs \((\sigma, \tau)\) of pairings of

\[
I = \{(t, p) : 1 \leq t \leq T, 1 \leq p \leq q\},
\]

and \(\delta_{\sigma}(i)\) is 1 if \(\sigma\) matches the same indices of \(i\) and 0 otherwise.

The expression of \([Wg](\sigma, \tau)\) is complicated but expanding on *Collins-Matsumoto (2017)* we have upper and lower bounds.
Haar unitary vs Gaussian

Let $G_{ij}$ be iid standard Gaussian variables.

**Theorem**

For $k = qT \leq n^{4/7}$,

$$n^{k/2} \left| \mathbb{E} \prod_{t=1}^{T} \left[ \prod_{p=1}^{q} U_{i_tp,j_tp} \right] \right| \leq (1 + \delta) \mathbb{E} \prod_{t=1}^{T} \left( \left[ \prod_{p=1}^{q} G_{i_tp,j_tp} \right] + \eta \right),$$

with $\delta = 3k^{7/2}n^{-2}$ and $\eta = 2k^q n^{-1/2}$.

The right-hand side can be estimated by using Wick calculus.
Concluding words
Further directions

Among open directions:

- non-asymptotic bounds \((\text{Bandeira-Boedihardjo-van Handel})\).
- tensor product of permutation matrices,
- random and deterministic unitary matrices,
- replace the free group by other non-amenable groups, such as surface groups \((\text{Magee, Naud, Puder})\) or free products of finite groups \((\text{Puder, Zimhoni})\).
- what happens for \(n\) fixed and \(q \to \infty\)?
Simulation

For \((U_1, \ldots, U_d)\) iid Haar on \(SU_n\) and

\[
\sum_{i=1}^{d} U_i \otimes q_i + U_i^* \otimes q_i.
\]
Simulation

For \( (U_1, \ldots, U_d) \) iid Haar on \( SU_n \) and

\[
\sum_{i=1}^{d} U_i^\otimes q + U_i^* \otimes q.
\]
Thank you for your attention!