Spherical convex hull of random points on a wedge

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supported by a grant from the National Science Foundation

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- the volume $V_d(K_n)$ of K_n, or, more generally $V_k(K_n)$, $k = 0, 1, \cdots, d$, the $k-th$ intrinsic volume of K_n
- the number $f_{d-1}(K_n)$ of facets of K_n, and, more generally the number $f_k(K_n)$ of k-dimensional faces of K_n, $k = \{0, 1, \ldots, d-1\}$

II. The Poisson point process model

K is a fixed convex body $K \subset \mathbb{R}^d$
η_γ is a Poisson point process in $\mathbb{R}^d$ with intensity measure

$$\gamma\,\lambda, \qquad \gamma > 0$$

λ is the Lebesgue measure, restricted to K

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Poisson random polytope K_{η_γ} is the convex hull of the Poisson point process η_γ

Note: the expected number of points of η_γ equals $\gamma V_d(K)$
$\Longrightarrow$
for a Poisson random polytope:
$\gamma V_d(K)$ plays the same role as n for the classical random polytopes

We will look at the **expected number of** k**-dimensional faces of** K_n

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The behavior depends on the geometry of the underlying convex body K

I. Euclidean space

- If K is C^2_+, then

$$\mathbb{E}f_k(K_n) \sim c_{d,k}\, as(K)\, n^{\frac{d-1}{d+1}} \qquad \text{as } n \to \infty \tag{1}$$

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  - $c_{d,k}$ is a constant only depending on d and on k and
  - $as(K) = \int_{\partial K} \kappa(K,x)^{1/(d+1)}\,\mathrm{d}x$ is the affine surface area of K
  - $\kappa(K,x)$ is the Gauss curvature at $x \in \partial K$

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  - (1) was proved, depending on k, by Bárány, Schütt, Reitzner, Wieacker

- If $K = P$ is a d-dimensional polytope, then

$$\mathbb{E} f_k(K_n) \sim \hat{c}_{d,k} \operatorname{flag}(P) (\log n)^{d-1} \qquad \text{as } n \to \infty \tag{2}$$

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$$F_0 \subset F_1 \subset \ldots \subset F_{d-1}$$

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(2) was proved by

- Rényi and Sulanke for $d = 2$
- Bárány and Buchta for general d and $k = 0, d-1$
- Reitzner for general d and k

II. Spherical space

Let $\mathbb{S}^d$ denote the unit sphere in $\mathbb{R}^{d+1}$ and

$$\mathbb{S}^d_+ := \mathbb{S}^d \cap \{x_{d+1} \geq 0\} \subset \mathbb{R}^{d+1}$$

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A set $K \subset \mathbb{S}^d \cap \{x_{d+1} > 0\}$ is a spherical convex body, if it is closed, and its positive hull

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σ_d denotes the spherical volume, i.e., the d-dimensional Hausdorff measure on $\mathbb{S}^d$

We consider random polytopes that are the spherical convex hull of points chosen uniformly according to $\frac{\sigma_d}{\sigma_d(K)}$ in K

- Analogue to (1)

Theorem (Besau, Ludwig, W)

Let $K \subset \mathbb{S}^d_+$ be a spherical convex body. If K_n is the spherical convex hull of n random points chosen uniformly in K, then

$$\lim_{n\to\infty} \mathbb{E} f_0(K_n)\, n^{-\frac{d-1}{d+1}} = \beta_d\, vol_d(K)^{-\frac{d-1}{d+1}} \int_{\partial K} \kappa^{\mathbb{S}^d}(K,x)^{\frac{1}{d+1}}\, dx$$

- $\beta_d = \frac{(d^2+d+2)(d^2+1)}{2(d+3)\cdot(d+1)!}\, \Gamma\left(\frac{d^2+1}{d+1}\right) \left(\frac{d+1}{|B_2^{d-1}|}\right)^{\frac{2}{d+1}}$
- $\kappa^{\mathbb{S}^d}(K,x)$ is the spherical Gauss-Kronecker curvature

Sketch of Proof

The gnomonic projection $g : \mathbb{S}^d_+ \to \mathbb{R}^d$ is defined by

$$g(x) = \bar{x} = \frac{x}{x \cdot e_{d+1}} - e_{d+1},$$

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- $\bar{K} = g(K)$ is a convex body in $\mathbb{R}^d$
- The pushforward of the measure σ_d under g is the measure $d\Phi = \psi \, d\bar{x}$ with density

$$\psi(\bar{x}) = \frac{1}{(1 + \|\bar{x}\|^2)^{(d+1)/2}}$$

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Choosing n points in K with respect to $\frac{\sigma_d}{\sigma_d(K)}$ corresponds to choosing n points in $\bar{K}$ with respect to

$$\frac{\psi(\bar{x}) \, d\bar{x}}{\int_{\bar{K}} \psi(\bar{x}) \, d\bar{x}},$$

i.e., $\quad g(K_n) = (g(K))_n^{\Phi} = \left(\bar{K}\right)_n^{\Phi}$

We use a result by Böröczky, Fodor and Hug:

• L be a convex body in $\mathbb{R}^d$
• $\phi : L \to (0, \infty)$ is a continuous probability density
• the random polytope L_n^Φ is the convex hull of n independent random points chosen in L according to the probability measure Φ

$$\lim_{n\to\infty} \mathbb{E} f_0(L_n^\Phi)\, n^{-\frac{d-1}{d+1}} = \beta_d \int_{\partial L} \kappa(L,x)^{\frac{1}{d+1}} \phi(x)^{\frac{d-1}{d+1}}\, dx,$$

where β_d is as above

As $n \to \infty$, with $\Phi(\bar{x}) = \frac{\psi(\bar{x})\, d\bar{x}}{\int_{\bar{K}} \psi(\bar{x})\, d\bar{x}}$

$$\begin{aligned} \mathbb{E} f_0(K_n)\, n^{-\frac{d-1}{d+1}} &= \mathbb{E} f_0\left(\left(\bar{K}\right)_n^\Phi\right)\, n^{-\frac{d-1}{d+1}} \\ &\sim \frac{\beta_d}{\left(\int_{\bar{K}} \psi(\bar{x})\, d\bar{x}\right)^{\frac{d-1}{d+1}}} \int_{\partial \bar{K}} \kappa(\bar{K},\bar{x})^{\frac{1}{d+1}}\, \psi(\bar{x})^{\frac{d-1}{d+1}}\, d\bar{x} \end{aligned}$$

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- $\kappa^{\mathbb{S}^d}(K, x) = \kappa(\bar{K}, \bar{x}) \left(\frac{1+\|\bar{x}\|^2}{1+(\bar{x} \cdot N_{\bar{K}}(\bar{x}))^2} \right)^{\frac{d+1}{2}}$

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$$J^{\partial \bar{K}} g^{-1}(\bar{x}) = \frac{\left(1 + (\bar{x} \cdot N_{\bar{K}}(\bar{x}))^2\right)^{\frac{1}{2}}}{(1 + \|\bar{x}\|^2)^{\frac{d}{2}}}$$

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- the Jacobian on $\bar{K}$ of g^{-1} is

$$J^{\bar{K}} g^{-1}(\bar{x}) = \left(1 + \|\bar{x}\|^2\right)^{-\frac{d+1}{2}}$$

Let $(X_i)_{i\geq 1}$ be a sequence of independent random points uniformly distributed on $\mathbb{S}^d_+ := \mathbb{S}^d \cap \{x_{d+1} \geq 0\} \subset \mathbb{R}^{d+1}$

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For $n \geq d+1$, let

$$\begin{aligned} K_n^{(s)} : &= [X_1, \cdots, X_n]_{\mathbb{S}^d} \\ &= \operatorname{pos}(X_1, \ldots, X_n) \cap \mathbb{S}^d \end{aligned}$$

be the spherical convex hull of the random points $X_1, \cdots, X_n$

$$\operatorname{pos}(X_1, \ldots, X_n) := \{\lambda_1 X_1 + \cdots + \lambda_n X_n \colon \lambda_1, \ldots, \lambda_n \geq 0\} \subset \mathbb{R}^{d+1}$$

Theorem (Bárány, Hug, Reitzner, Schneider $k \in \{0, d-1\}$; Kabluchko, Marynych, Temesvari, Thäle, general k)

$$\lim_{n\to\infty} \mathbb{E} f_k(K_n^{(s)}) = \tilde{c}_{d,k}, \tag{3}$$

where $\tilde{c}_{d,k}$ *is a constant only depending on* d *and* k

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Applying the gnomonic projection g

- to the "spherical convex polytope" $\mathbb{S}^d_+$, $\mathbb{R}^d$ can be seen as a d-dimensional convex "unbounded polytope" with a single facet at ∞

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Applying the gnomonic projection g

- to the "spherical convex polytope" $\mathbb{S}_+^d$, $\mathbb{R}^d$ can be seen as a d-dimensional convex "unbounded polytope" with a single facet at ∞
- $K_n^{(s)}$ is identified with the convex hull of n random points chosen w. r. to the normalized pushforward of σ_d, $\frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}}\psi(\bar{x})\, d\bar{x} = \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}} \frac{d\bar{x}}{(1+\|\bar{x}\|^2)^{(d+1)/2}}$

Compare behavior of **bounded** polytopes in $\mathbb{R}^d$ with the **'unbounded polytope'** we ask:

Are there models for random polytopes that interpolate for $\mathbb{E} f_k(K_n)$ *between the behavior of*

$$\hat{c}_{d,k}\, \mathrm{flag}(P)(\log n)^{d-1} \quad \text{and} \quad \tilde{c}_{d,k}$$

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- We will look at $k = d - 1$

SETTING

Let $j \in \{1, \ldots, d\}$
Let $H_1, \ldots, H_j$ be distinct hyperplanes passing through the origin of $\mathbb{R}^{d+1}$
Let
$$\mathbb{S}^d_{j,+} := \mathbb{S}^d \cap H_1^+ \cap \ldots \cap H_j^+,$$
H_i^+ is the positive halfspace, bounded by the hyperplane H_i, $i \in \{1, \ldots, j\}$

- $\mathbb{S}^d_{j,+}$ is a d-dimensional spherical convex subset of $\mathbb{S}^d$
- its shape is determined by the angles between $H_1, \ldots, H_j$

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Let $(X_i)_{i \geq 1}$ be independent random points uniformly distributed on $\mathbb{S}^d_{j,+}$

For $n \geq d+1$, let $K_n^{(s,j)}$ be the spherical convex hull of $X_1, \ldots, X_n$

Conjecture. *For $j \in \{1, \dots, d\}$ one has that*

$$\mathbb{E} f_{d-1}(K_n^{(s,j)}) \sim c_{d,j} \, (\log n)^{j-1} \qquad \text{as } n \to \infty$$

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In particular, we conjecture

- the first-order asymptotic expansion does not depend on the angles between $H_1, \dots, H_j$
- the error terms depend on the angles between the hyperplanes $H_1, \cdots, H_j$

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We prove the conjecture in the case $j = 2$ and under the assumption that the angle $\alpha(H_1, H_2)$ between the hyperplanes H_1 and H_2 is a right angle.

We call the set $\mathbb{S}^d_{2,+}$ a spherical wedge

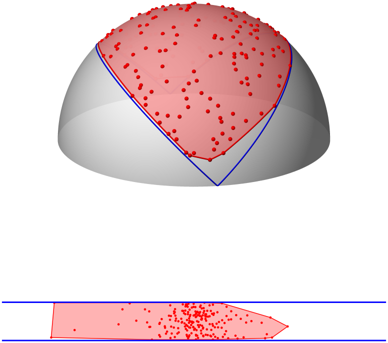

Figure: The upper panel shows a random spherical polygon in the spherical wedge of dimension two. The same random spherical polygon is shown in the lower panel after gnomonic projection in the center of the spherical wedge.

Main Theorem

Theorem (Besau, Gusakova, Reitzner, Schütt, Thäle, W)

Let $K = \mathbb{S}^d_{2,+}$ *and suppose that* $\alpha(H_1, H_2) = \frac{\pi}{2}$. *Then there exists a constant* $c_{d,2} > 0$ *only depending on the dimension* d *such that*

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Remarks

- $c_{d,2} = \frac{2^{d-1}}{d} |\partial B_2^d| \, A_d, \qquad c_{2,2} = \frac{4}{3}$

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- in dimension 2: $\mathbb{E} f_1(K_n^{(s,2)}) \sim \frac{4}{3} (\log n) \qquad$ as $n \to \infty$

- in dimension 2: $\mathbb{E}f_1(K_n^{(s,2)}) \sim \frac{4}{3}(\log n)$ as $n \to \infty$

Let $K_n^{(\ell)}$ be the convex hull of n independent and uniform random points in a planar polygon with $\ell \geq 3$ edges

- Rényi and Sulanke: $\mathbb{E}f_1(K_n^{(\ell)}) \sim \frac{2\ell}{3}(\log n)$ as $n \to \infty$

Steps of the proof

$$
\begin{aligned}
&\mathbb{E} f_{d-1}(K_n^{(s,2)}) \\
&= \mathbb{E} \sum_{1 \le i_1 < \cdots < i_d \le n} \mathbf{1}\{\mathbf{x}_{i_1}, \cdots, \mathbf{x}_{i_d} \text{ generate a facet of } K_n^{(s,2)}\} \\
&= \binom{n}{d} \int_{\mathbb{S}^d_{2,+}} \cdots \int_{\mathbb{S}^d_{2,+}} \mathbb{P}(\mathbf{x}_1, \cdots, \mathbf{x}_d \text{ generate a facet of } K_n^{(s,2)}) \frac{\sigma_d(d\mathbf{x}_1)}{\sigma_d(\mathbb{S}^d_{2,+})} \cdots \frac{\sigma_d(d\mathbf{x}_d)}{\sigma_d(\mathbb{S}^d_{2,+})}
\end{aligned}
$$

$$\mathbb{E} f_{d-1}(K_n^{(s,2)}) =$$

$$\binom{n}{d} \int_{\mathbb{S}^d_{2,+}} \cdots \int_{\mathbb{S}^d_{2,+}} \mathbb{P}\left(\mathbf{x}_1, \cdots, \mathbf{x}_d \text{ generate a facet of } K_n^{(s,2)}\right) \frac{\sigma_d(\mathrm{d}\mathbf{x}_1)}{\sigma_d(\mathbb{S}^d_{2,+})} \cdots \frac{\sigma_d(\mathrm{d}\mathbf{x}_d)}{\sigma_d(\mathbb{S}^d_{2,+})}$$

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Spherical Blaschke-Petkantschin Formula [Bárány, Hug, Reitzner, Schneider]

$$\int_{\mathbb{S}^d} \cdots \int_{\mathbb{S}^d} f(\mathbf{x}_1, \ldots, \mathbf{x}_d)\, \sigma_d(\mathrm{d}\mathbf{x}_1) \ldots \sigma_d(\mathrm{d}\mathbf{x}_d) = \frac{\omega_{d+1}}{2} \times$$

$$\int_{G(d+1,d)} \left[\int_{\mathbb{S}^d \cap H} \cdots \int_{\mathbb{S}^d \cap H} f(\mathbf{x}_1, \ldots, \mathbf{x}_d) \times \nabla_d(\mathbf{x}_1, \ldots, \mathbf{x}_d)\, \sigma_{d-1}(\mathrm{d}\mathbf{x}_1) \ldots \sigma_{d-1}(\mathrm{d}\mathbf{x}_d)\right] \nu_d(\mathrm{d}H)$$

- $f : \mathbb{S}^d \to \mathbb{R}$ is a Borel measurable function, $\omega_{d+1} = \sigma_d(\mathbb{S}^d)$
- $G(d+1, d)$ is the Grassmannian of d-dimensional linear subspaces of $\mathbb{R}^{d+1}$ with the rotation invariant Haar probability measure ν_d
- $\nabla_d(\mathbf{x}_1, \ldots, \mathbf{x}_d)$ is the Euclidean volume of the d-dimensional parallelotope spanned by $\mathbf{x}_1, \ldots, \mathbf{x}_d$

$\mathbb{P}(\mathbf{x}_1, \cdots, \mathbf{x}_d$ generate a facet of $K_n^{(s,2)})$ happens if :

- d points $x_1 \cdots x_d$ are chosen in $\mathbb{S}^d_{2,+} \cap H$
- $n-d$ points are chosen in either $\mathbb{S}^d_{2,+} \cap H^+$ or $\mathbb{S}^d_{2,+} \cap H^-$
  $\longrightarrow \left(\frac{\sigma_d(\mathbb{S}^d_{2,+} \cap H^+)}{\sigma_d(\mathbb{S}^d_{2,+})}\right)^{n-d} + \left(\frac{\sigma_d(\mathbb{S}^d_{2,+} \cap H^-)}{\sigma_d(\mathbb{S}^d_{2,+})}\right)^{n-d}$

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$$\begin{aligned}\mathbb{E} f_{d-1}(K_n^{(s,2)}) &= \frac{\omega_{d+1}}{2\sigma_d(\mathbb{S}^d_{2,+})^d}\binom{n}{d}\int_{G(d+1,d)} \\ &\times \left[\int_{\mathbb{S}^d_{2,+}\cap H} \cdots \int_{\mathbb{S}^d_{2,+}\cap H} \nabla_d(\mathbf{x}_1, \cdots, \mathbf{x}_d)\, \sigma_{d-1}(\mathrm{d}\mathbf{x}_1)\cdots\sigma_{d-1}(\mathrm{d}\mathbf{x}_d)\right] \\ &\times \left[\left(\frac{\sigma_d(\mathbb{S}^d_{2,+} \cap H^+)}{\sigma_d(\mathbb{S}^d_{2,+})}\right)^{n-d} + \left(\frac{\sigma_d(\mathbb{S}^d_{2,+} \cap H^-)}{\sigma_d(\mathbb{S}^d_{2,+})}\right)^{n-d}\right] \nu_d(\mathrm{d}H)\end{aligned}$$

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$$\frac{\omega_{d+1}}{2\sigma_d(\mathbb{S}^d_{2,+})^d}\binom{n}{d}\int_{G(d+1,d)}\left[\int_{\mathbb{S}^d_{2,+}\cap H}\cdots\int_{\mathbb{S}^d_{2,+}\cap H}\nabla_d(\mathbf{x}_1,\cdots,\mathbf{x}_d)\,\sigma_{d-1}(\mathrm{d}\mathbf{x}_1)\cdots\sigma_{d-1}(\mathrm{d}\mathbf{x}_d)\right]$$

$$\times\left[\left(\frac{\sigma_d(\mathbb{S}^d_{2,+}\cap H^+)}{\sigma_d(\mathbb{S}^d_{2,+})}\right)^{n-d}+\left(\frac{\sigma_d(\mathbb{S}^d_{2,+}\cap H^-)}{\sigma_d(\mathbb{S}^d_{2,+})}\right)^{n-d}\right]\nu_d(\mathrm{d}H)$$

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$$\times\left[\left(\frac{\sigma_d(\mathbb{S}^d_{2,+}\cap H^+)}{\sigma_d(\mathbb{S}^d_{2,+})}\right)^{n-d}+\left(\frac{\sigma_d(\mathbb{S}^d_{2,+}\cap H^-)}{\sigma_d(\mathbb{S}^d_{2,+})}\right)^{n-d}\right]\nu_d(\mathrm{d}H)$$

$$=\frac{1}{\sigma_d(\mathbb{S}^d_{2,+})^d}\binom{n}{d}\int_{\mathbb{S}^d}\left[\int_{\mathbb{S}^d_{2,+}\cap H(\mathbf{z})}\cdots\int_{\mathbb{S}^d_{2,+}\cap H(\mathbf{z})}\nabla_d(\mathbf{x}_1,\ldots,\mathbf{x}_d)\,\sigma_{d-1}(\mathrm{d}\mathbf{x}_1)\ldots\sigma_{d-1}(\mathrm{d}\mathbf{x}_d)\right]$$

$$\times\left(\frac{\sigma_d(\mathbb{S}^d_{2,+}\cap H^-)}{\sigma_d(\mathbb{S}^d_{2,+})}\right)^{n-d}\sigma_d(dz)$$

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expected d-dimensional volume of the parallelopiped spanned by $(U_i, \mathbf{Z}_i, 1)$, $1 \le i \le d$

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- $U_1, \ldots, U_d$ are random variables uniformly distributed on $[-1, 1]$
- $\mathbf{Z}_1, \ldots, \mathbf{Z}_d$ are random vectors distributed according to a beta-prime distribution on $\mathbb{R}^{d-2}$ with parameter $\beta = \frac{d+1}{2}$ and probability density function

$$\frac{\Gamma(\beta)}{\pi^{\frac{d-2}{2}} \Gamma(\beta - \frac{d-2}{2})} \left(1 + \|\mathbf{x}\|^2\right)^{-\beta}$$

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$$A_2 = \int_{-1}^{1} \int_{-1}^{1} |x - y| \, \frac{dx}{2} \, \frac{dy}{2} = \frac{2}{3}$$