Submodularity in Convex Geometry

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Submodularity : Definition

Let $[M] = \{1, 2, \ldots, M\}$ and $F : 2^{[M]} \to \mathbb{R}$ (or $\overline{\mathbb{R}}$).

We say that F is <u>submodular</u> if, $\forall\ s, t \subset [M]$,

$$F(s \cup t) + F(s \cap t) \leq F(s) + F(t).$$

If $-F$ is submodular, we say F is <u>supermodular</u>.

<u>E.g:</u> 1) Cardinality $F(s) = |s|$ is submodular and supermodular.

2) Rank function of a matroid is submodular.

3) If $X_1, \ldots, X_M$ are discrete r.v's, and $X_s = (X_i : i \in s)$, then $F(s) = H(X_s)$ is submodular, where H is entropy.

Why study submodularity?

- Implies many other inequalities:

Fact (Moulin-Ollagnier & Pinchon '82, Madiman - Tetali '10):

If F submodular and $F(\phi) = 0$, then F is fractionally subadditive

E.g.: $$F([M]) \leq \frac{1}{M-1} \sum_{|S|=M-1} F(S) \leq \frac{1}{\binom{M-1}{k-1}} \sum_{|S|=k} F(S) \leq \sum_{i=1}^{M} F(\{i\}).$$

- Submodular : discrete :: convex : continuous.

There are efficient algorithms in combinatorial optimization for optimizing submodular functions + they arise naturally in applications

An observation for Convex Bodies

Fact: (Fradelizi - Marsiglietti - M. - Zvavitch 2018, but maybe folklore?)

For compact, convex sets $A_1, \ldots, A_M \subset \mathbb{R}^n$,

$$F_V(S) = \mathrm{Vol}_n\left(\sum_{i \in S} A_i\right) = \left|\sum_{i \in S} A_i\right| \quad \text{is supermodular.}$$

Rmks: 1) Note that supermodularity of F_V is equivalent to

$$|A+B+C| + |A| \geq |A+B| + |A+C|.$$

Take $A = \sum_{i \in S \cap t} A_i$, $B = \sum_{i \in S \setminus t} A_i$, $C = \sum_{i \in t \setminus S} A_i$.

2) Easily proved using properties of mixed volumes: we will show something more general later.

Mixed Volumes

Fact: If $A_1, \dots, A_M \in \mathcal{K}_n$, then for $x = (x_1, \dots, x_M) \in \mathbb{R}_+^M$,

$$\left| \sum_{i=1}^{M} x_i A_i \right| = \sum_{\substack{\alpha = (\alpha_1, \dots, \alpha_M): \\ \sum \alpha_i = n}} V(A_1[\alpha_1], \dots, A_M[\alpha_M]) \, x_1^{\alpha_1} \cdots x_M^{\alpha_M}$$

is a homogeneous polynomial in the M variables (x) of degree n.

Notation: $A_i[\alpha_i]$ means A_i appears α_i times (possibly 0), so there are always exactly n sets in $V(\dots)$

Properties:

1) $V(A_1, \dots, A_n) \geq 0$, with equality iff
$$\exists \, \phi \neq S \subset [n] \text{ s.t. } \dim\left(\sum_{i \in S} A_i\right) < \# S.$$

2) V is multilinear, symmetric, and translation-invariant.

3) $V(A, \dots, A) = |A|$, $nV(A[n-1], B_2^n) = |\partial A|$, etc.

A Question in the Opposite Direction

Background:

Additive Combinatorics: The Plünnecke-Ruzsa inequality says that if A, B_1, B_2 are finite subsets of an abelian group G, and $\#(A+B_i) \leq c_i \cdot \#(A)$, then $\exists\, X \subset A$ s.t. $\#(X+B_1+B_2) \leq c_1 c_2 \#(X)$.

Q: Can we take $X = A$ if all our sets are convex?

Information Theory: If X, Y_1, Y_2 are independent r.v's taking values in a LCA group G, $h(X+Y_1+Y_2) + h(X) \leq h(X+Y_1) + h(X+Y_2)$.
[Madiman '08]

Suggests the Question: Is $|A+B_1+B_2| \cdot |A| \leq |A+B_1| \cdot |A+B_2|$

$\forall\, A, B_1, B_2 \in \mathcal{K}_n$?

Rmk: Bobkov-M. 2012 showed this with a constant 3^n on the right.

Log-submodularity of Volume : Why study?

Theorem: Let $\mathcal{C} \subset \mathcal{K}_n$ be invariant under dilation and sums. The following are equivalent: [Fradelizi-M.-Meyer-Zvavitch 2022]

(1) For $A_1, \dots, A_m \in \mathcal{C}$, $F(S) = \left|\sum_{i \in S} A_i\right|$ is log-submodular.

(2) $|A + B_1 + B_2| \cdot |A| \leq |A + B_1| \cdot |A + B_2| \quad \forall\ A, B_1, B_2 \in \mathcal{C}$.

(3) $|A| \cdot V(A[n-2], B_1, B_2) \leq \frac{n}{n-1} V(A[n-1], B_1)\, V(A[n-1], B_2), \forall\ A, B_1, B_2 \in \mathcal{C}$

Rmk: Let $A, B_1, B_2 \in \mathcal{K}_n$. Then:

Alexandrov-Fenchel inequality: $|A| \cdot V(A[n-2], B[2]) \leq V(A[n-1], B[1])^2$.

Local A-F ineq: $|A| \cdot V(A[n-2], B_1, B_2) \leq 2 \cdot V(A[n-1], B_1) \cdot V(A[n-1], B_2)$

[Fenchel 1936, Giannopoulos-Hartzoulaki-Paouris 2002, Fradelizi-Giannopoulos-Meyer 2003, Dembo-Cover-Thomas '91, Artstein-Florentin-Ostrover 2014, Soprunov-Zvavitch 2016, ...]

Consequences of the Equivalence Theorem

In $|A| \cdot V(A[n-2], B_1, B_2) \leq 2 \cdot V(A[n-1], B_1) \cdot V(A[n-1], B_2)$ for $A, B_1, B_2 \in \mathcal{K}_n$, the constant 2 is sharp in every dimension $n \geq 2$, thanks to Giannopoulos-Hartzoulaki-Paouris 2002.

Since this inequality with $\frac{n}{n-1}$ in place of 2 is equivalent to log-submodularity of volume, we immediately have:

- Log-submodularity holds on $\mathcal{K}_2$
- Log-submodularity fails on $\mathcal{K}_3$ [Nayar-Tkocz 2017, personal communication]

Proof Ideas

Def^n: $f : \mathbb{R}_+^M \to \mathbb{R}$ is submodular if

$$f(x \wedge y) + f(x \vee y) \leq f(x) + f(y) \quad \forall\, x, y \in \mathbb{R}_+^M.$$

Note f submodular $\Longrightarrow$ $F(S) = f(\mathbb{1}_S) = f\big|_{\{0,1\}^M}$ is a submodular set function.

Lemma: If $f : \mathbb{R}_+^M \to \mathbb{R}$ is C^2, f is submodular $\Longleftrightarrow$ $\partial^2_{i,j} f \leq 0$ $\forall\, i \neq j$, $\forall x \in \mathbb{R}_+^M$

Pf: Consider $w(x) = \left| \sum_{i=1}^M x_i A_i \right| = |K_x|$, say.

Then $\partial_j w(x) = \lim\limits_{\varepsilon \to 0} \dfrac{|K_x + \varepsilon A_j| - |K_x|}{\varepsilon} = n \cdot V(K_x[n-1], A_j)$

and for $k \neq j$, $\partial^2_{j,k} w(x) = n(n-1)\, V(K_x[n-2], A_j, A_k)$

Note $f = \log w$ submodular $\Longleftrightarrow$ $\dfrac{w \cdot \partial_{jk} w - \partial_j w \cdot \partial_k w}{w^2} \leq 0$.

Log-submodularity of Volume : More Equivalences

Theorem: Let $\mathcal{C} \subset K_n$ be invariant under dilation, Minkowski sums, and linear transformations. The following are equivalent: [FMMZ 2022]

(1) For $A_1, \dots, A_M \in \mathcal{C}$, $F(S) = \left|\sum_{i \in S} A_i\right|$ is log-submodular.

(2) $|A + B_1 + B_2| \cdot |A| \leq |A + B_1| \cdot |A + B_2| \quad \forall\ A, B_1, B_2 \in \mathcal{C}$.

AFO 2014

(3) $|A| \cdot V(A[n-2], B_1, B_2) \leq \frac{n}{n-1} V(A[n-1], B_1)\, V(A[n-1], B_2), \forall\ A, B_1, B_2 \in \mathcal{C}$.

(4) $\dfrac{|A|}{|\partial A|} \leq \dfrac{|P_{u^\perp} A|}{|\partial (P_{u^\perp} A)|} \quad \forall\ A \in \mathcal{C}, u \in S^{n-1}$.

(5) $|A| \cdot |P_{\{u,v\}^\perp} A| \cdot \sqrt{1 - \langle u, v\rangle^2} \leq |P_{u^\perp} A| \cdot |P_{v^\perp} A|, \ \forall\ A \in \mathcal{C}, u, v \in S^{n-1}$.

(6) For any $A \in \mathcal{C}$ and $u, v \in \mathbb{R}^n$, $P(t) = |A + t([0,u] + [0,v])|$ is the restriction to $\mathbb{R}_+$ of a polynomial on $\mathbb{R}$ with only real roots.

Zonoids

Zonotopes: A zonotope is a Minkowski sum of finitely many line segments.

By translation invariance, a zonotope is of form $\sum_{i=1}^{k} [0, u_i]$, for $u_i \in \mathbb{R}^n$, and $k \in \mathbb{N}$. If $k \leq n$, this is a parallelotope; if $k > n$, it can be more complicated.

Zonoid: A limit (in Hausdorff metric) of zonotopes.

Every face of a zonoid is (centrally) symmetric. (E.g: $B_2^n \in \mathcal{Z}_n$)

Facts: 1) $\det\left((u_i)_{i=1,\dots,n}\right) = \left| \sum_{i=1}^{n} [0, u_i] \right| = n!\, V\left([0,u_1], \dots, [0,u_n]\right)$

2) If $M > n$, $\left| \sum_{i=1}^{M} [0, u_i] \right| = \frac{1}{n!} \sum_{|S|=n} \det\left((u_i)_{i \in S}\right)$.

Log-Submodularity Conjecture for Zonoids

Conjecture: If $A, B, C \in \mathcal{Z}_n$, $|A+B+C| \cdot |A| \leq |A+B| \cdot |A+C|$.

Theorem: Conjecture is true for $n=3$.

$$|A|_3 \cdot |P_{\{e_1,e_2\}^\perp} A|_1 \leq |P_{e_1^\perp} A|_2 \cdot |P_{e_2^\perp} A|_2 \iff \sum_{\{i,j,k\}} \left| \det \begin{pmatrix} x_i & x_j & x_k \\ y_i & y_j & y_k \\ z_i & z_j & z_k \end{pmatrix} \right| \sum_{i=1}^{M} |z_i|$$

for $A = \sum_{i=1}^{M} [0, u_i]$, $u_i = \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}$

$$\leq \sum_{\{i,j\}} \left| \det \begin{pmatrix} x_i & x_j \\ z_i & z_j \end{pmatrix} \right| \sum_{\{i,j\}} \left| \det \begin{pmatrix} y_i & y_j \\ z_i & z_j \end{pmatrix} \right|$$

Non-trivial determinant inequality.

Rmk: An even stronger statement holds in $\mathbb{R}^2$ when $A =$ Euclidean ball.

Courtade conjectured $|A|^{\frac{1}{n}} |A+B+C|^{\frac{1}{n}} + |B|^{\frac{1}{n}} |C|^{\frac{1}{n}} \leq |A+B|^{\frac{1}{n}} |A+C|^{\frac{1}{n}}$ when $A = B_2^n$. This is true for $n=2$.

Proof of k-supermodularity

Def^n: $F : 2^{[M]} \to \mathbb{R}$ is k-supermodular if $\forall$ s_0 and disjoint $s_1, \dots, s_k \subset [M]$,

$$\sum_{I \subset [k]} (-1)^{k-|I|} F\Big(s_0 \cup \{\bigcup_{i \in I} s_i\}\Big) \geq 0$$

Rmks: $k=1$: $F(s \cup t) \geq F(s)$

$k=2$: $F(s \cup t) + F(s \cap t) \geq F(s) + F(t)$

$$\Leftrightarrow F(s \cup t) - F(s) \geq F(t) - F(s \cap t)$$

Iterated "differences" are "increasing".

Idea: Apply to $F(s) = \Big|\sum_{i \in s} A_i\Big|$ and $f(x) = \Big|\sum_{i \in [M]} x_i A_i\Big|$.

Th^m: If $f : \mathbb{R}_+^M \to \mathbb{R}$ is C^k, f is k-supermodular $\Leftrightarrow \partial^k_{i_1 \dots i_k} f \geq 0$ $\forall$ distinct $i_1, \dots, i_k \in [M]$.

In Summary...

Consider
$$F(S) = \left| \sum_{i \in S} A_i \right| : 2^{[M]} \to \mathbb{R}_+ ,$$
for fixed compact sets $A_1, \ldots, A_M$.

	Compact sets $\mathcal{B}_n$	Convex sets $\mathcal{K}_n$	Zonoids $\mathcal{Z}_n$	
Supermodularity	NO, even for $n=1$	YES	(YES)	FMMZ 2018
Fractional superadditivity	YES	(YES)	(YES)	Barthe-M. 2021
Log-submodularity	NO, even for $n=1$	YES for $n=2$. NO, for $n \geq 3$	YES for $n=3$. Conj. for $n \geq 4$.	FMMZ 2022

Merci beaucoup!

Extras

Supermodularity beyond convex sets?

Conjecture: [Fradelizi-M.-Marsiglietti-Zvavitch 2018]

If $A \in \mathcal{K}_n$ and $B, C \in \mathcal{B}_n$, then

$$|A+B+C| + |A| \geq |A+B| + |A+C|.$$

Rmks: 1) On $\mathbb{R}$, even stronger fact is true: if $A, B, C \in \mathcal{B}_1$,

$$|A+B+C| + |\mathrm{conv}(A)| \geq |A+B| + |A+C|$$

2) Fradelizi-M.-Zvavitch 2022 show that the Conjecture is true if B is a zonoid.

Connection to Brunn-Minkowski

Let $\mathcal{B}_n = \{\text{compact sets in } \mathbb{R}^n\}$, $\mathcal{K}_n = \{\text{convex, compact sets in } \mathbb{R}^n\}$.

Set $F_{BM}(s) = \left|\sum_{i \in s} A_i\right|^{\frac{1}{n}}$ for $s \subset [M]$, $A_1, \dots, A_M \in \mathcal{B}_n$.

Rmks:

1) Brunn-Minkowski inequality: F_{BM} is superadditive on $\mathcal{B}_n$.

2) Bobkov-M.-Wang 2011 noted that F_{BM} is frac. superadditive on $\mathcal{K}_n$.

Q: Is F_{BM} supermodular on $\mathcal{K}_n$? No, for $n \geq 2$.

Counterexample from Ghassemi-M. 2019 to supermodularity of $\det^{\frac{1}{n}}$:

$$A_1 = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad A_2 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

$\sqrt{\det(A_1+A_2+A_3)} = 2.6$, $\sqrt{\det(A_1+A_2)} = 2.5$, $\sqrt{\det(A_1+A_3)} = \sqrt{2.1} \simeq 1.45$

Refined Brunn-Minkowski ?

Q: Is F_{BM} fractionally superadditive on $\mathcal{B}_n$?

Bobkov-M.-Wang 2011 conjectured YES based on various evidences, including that it is true on $\mathcal{K}_n$.

A: YES, for $n=1$.

Barthe-M. 2021

No, in general.

Fradelizi-M.-Marsiglietti-Zvavitch 2016 for $n \geq 12$

Fradelizi-Langi-Zvavitch 2020 for $n \geq 7$

Still open for $2 \leq n \leq 6$.

Summary for F_{BM}

Consider $\quad F_{BM}(S) = \left| \sum_{i \in S} A_i \right|^{\frac{1}{n}} : 2^{[M]} \longrightarrow \mathbb{R}_+$,

for fixed compact sets $A_1, \dots, A_M \subset \mathbb{R}^n$.

	Compact sets $\mathcal{B}_n$	Convex sets $\mathcal{K}_n$	Zonoids $\mathcal{Z}_n$
Supermodularity	FMMZ'18 NO, even for $n=1$	YES, for $n=1$ (NO, for $n \geq 2$)	(YES, for $n=1$) NO, for $n \geq 2$. Ghassemi-M.'19
Fractional superadditivity	YES, for $n=1$ NO, for $n \geq 7$ Barthe-M.'21 + FMMZ'16 + FLZ'20	YES Bobkov-M.-Wang'11	(YES)

Note: Log-submodularity of F_{BM} and F are the same.

Best constants in Plünnecke-Ruzsa inequality on K_n

Set
$$c_n = \sup_{A,B,C \in K_n} \frac{|A+B+C| \cdot |A|}{|A+B| \cdot |A+C|}.$$

Note $F(S) = \left|\sum_{i \in S} A_i\right|$ is log-submodular on $K_n \Leftrightarrow c_n \leq 1$.

What is known?

- $c_2 = 1$
- $c_3 = \frac{4}{3}$
- $c_4 \leq 2$
- For large n, $\left(\frac{4}{3} + o(1)\right)^n \leq c_n \leq \left(\frac{1+\sqrt{5}}{2}\right)^n \simeq 1.618^n$.

Nayar–Tkocz 2017 (personal communication), FMZ 2022