$L_p$ Steiner formula and its coefficients

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Phenomena in High Dimension
Classical Steiner formula

For a convex body $K$ and $t \geq 0$, the classical Steiner formula is

$$\text{Vol}_n(K + tB_2^n) = \sum_{i=0}^{n} \binom{n}{i} W_i(K) t^i = \sum_{i=0}^{n} \text{Vol}_{n-i}(B_2^{n-i}) V_i(K) t^{n-i},$$

where $B_2^n$ is a Euclidean unit ball in $\mathbb{R}^n$. 

![Diagram of a convex body $K$ and its Steiner transformation $K + tB_2^n$.]
For a convex body $K$ and $t \geq 0$,

$$\text{Vol}_n(K + tB_2^n) = \sum_{i=0}^{n} \binom{n}{i} W_i(K) t^i = \sum_{i=0}^{n} \text{Vol}_{n-i}(B_2^{n-i}) V_i(K) t^{n-i}.$$ 

The coefficients $W_i(K)$ and $V_i(K)$ are called quermassintegrals and intrinsic volumes, respectively.

In particular,

$$W_0(K) = \text{Vol}_n(K) \quad \text{the volume}$$

$$W_1(K) = \frac{1}{n} \text{Vol}_{n-1}(\partial K) \quad \text{the surface area}$$

$$W_n(K) = \text{Vol}_n(B_2^n) \quad \text{the volume of the unit ball}$$
The radial function of a star body $K$ is defined by

$$\rho_K(u) = \max\{\lambda \geq 0 \mid \lambda u \in K\} \quad \text{for any } u \in S^{n-1}.$$  

For star bodies $K$ and $L$, the radial sum $\alpha K + \beta L$ is the star body with the radial function

$$\rho_{\alpha K + \beta L}(u) = \alpha \rho_K(u) + \beta \rho_L(u).$$
The radial function of a star body $K$ is defined by
\[
\rho_K(u) = \max\{\lambda \geq 0 \mid \lambda u \in K\} \text{ for any } u \in S^{n-1}.
\]

For star bodies $K$ and $L$, the radial sum $\alpha K \tilde{+} \beta L$ is the star body with the radial function
\[
\rho_{\alpha K \tilde{+} \beta L}(u) = \alpha \rho_K(u) + \beta \rho_L(u).
\]
For a star body $K$ and $t \geq 0$, we have that

$$\text{Vol}_n(K \tilde{+} tB_2^n) = \sum_{i=0}^{n} \binom{n}{i} \widehat{W}_i(K)t^i,$$

where $\tilde{+}$ is a radial addition. The coefficients $\widehat{W}_i(K)$ are called dual quermassintegrals that were introduced by Lutwak.
For a star body $K$ and $t \geq 0$, we have that

$$\text{Vol}_n(K \tilde{+} tB_2^n) = \sum_{i=0}^{n} \binom{n}{i} \tilde{W}_i(K) t^i,$$

where $\tilde{+}$ is a radial addition. The coefficients $\tilde{W}_i(K)$ are called dual quermassintegrals that were introduced by Lutwak.

Dual quermassintegrals of order $i$ can be written as

$$\tilde{W}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K(u)^{n-i} d\sigma(u).$$
For real $p \neq -n$, the $L_p$ affine surface area is defined as

$$as_p(K) = \int_{\partial K} \frac{H_{n-1}(x)^{\frac{p}{n+p}}}{\langle x, N(x) \rangle^{\frac{n(p-1)}{n+p}}} d\mathcal{H}^{n-1}(x),$$

where $H_{n-1}(x)$ is the Gauss curvature of $K$ at $x \in \partial K$ and $N(x)$ is the outer normal vector at $x$. When $p = 1$, we recover the classical affine surface area as

$$as_1(K) = \int_{\partial K} H_{n-1}(x)^{\frac{1}{n+1}} d\mathcal{H}^{n-1}(x).$$

For $p = 0$, we have

$$as_0(K) = \int_{\partial K} \langle x, N(x) \rangle d\mathcal{H}^{n-1}(x) = n \text{Vol}(K).$$
For real $p \neq -n$, the $L_p$ affine surface area is defined as

$$as_p(K) = \int_{\partial K} \frac{H_{n-1}(x)^{\frac{p}{n+p}}}{\langle x, N(x) \rangle^{\frac{n(p-1)}{n+p}}} d\mathcal{H}^{n-1}(x),$$

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When $p = 1$, we recover the classical affine surface area $as_1(K)$:

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$$as_1(K) = \int_{\partial K} H_{n-1}(x)^{\frac{1}{n+1}} d\mathcal{H}^{n-1}(x).$$

For $p = 0$,

$$as_0(K) = \int_{\partial K} \langle x, N(x) \rangle d\mathcal{H}^{n-1}(x) = n \text{Vol}_n(K).$$
If the boundary of $K$ is sufficiently smooth, then

$$a_{sp}(K) = \int_{S^{n-1}} f_K(u)^{\frac{n}{n+p}} h_K(u)^{\frac{n(1-p)}{n+p}} d\sigma(u),$$

where $f_K(u)$ is the curvature function, i.e. $f_K(u)$ is the reciprocal of $H_{n-1}(x)$ at $x \in \partial K$ that has $u$ as outer normal, and $h_K(u)$ is the support function of $K$. 
If the boundary of $K$ is sufficiently smooth, then

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For $p = \pm\infty$

$$as_{\pm\infty}(K) = \int_{S^{n-1}} \frac{1}{h_K(u)^n} \, d\sigma(u) = n \text{Vol}_n(K^\circ),$$

where $K^\circ = \{ y \in \mathbb{R}^n, \langle x, y \rangle \leq 1, \forall x \in K \}$ is the polar body of $K$. 
Theorem (T., Werner, 2019)

Let $K$ be a convex body in $\mathbb{R}^n$ that is $C^2$. Let $t \in \mathbb{R}$ be such that $0 \leq t < \min_{u \in S^{n-1}} h_K(u)$.

For all $p \in \mathbb{R}$, $p \neq -n$,

$$a_{sp}(K + tB^n_2) = \sum_{k=0}^{\infty} \left[ \sum_{m=0}^{k} \left( \frac{n(1-p)}{n+p} \right) \mathcal{W}_{m,k}^p(K) \right] t^k = \sum_{k=0}^{\infty} \mathcal{V}_k^p(K)t^k.$$  

In particular,

$$as_1(K + tB^n_2) = \sum_{k=0}^{\infty} \mathcal{W}_{k,k}(K)t^k = \sum_{k=0}^{\infty} \mathcal{V}_k^1(K)t^k.$$  

The coefficients $\mathcal{W}_{m,k}^p(K)$ and $\mathcal{V}_k^p(K)$ are called $L_p$ Steiner coefficients and $L_p$-Steiner quermassintegrals.
The \( j \)-th normalized elementary symmetric functions of the principal curvatures are:

\[
H_j = \left( \frac{n-1}{j} \right)^{-1} \sum_{1 \leq i_1 < \cdots < i_j \leq n-1} k_{i_1} \cdots k_{i_j}
\]

for \( j = 1, \ldots, n-1 \) and \( H_0 = 1 \).
The $j$-th normalized elementary symmetric functions of the principal curvatures are

$$H_j = \binom{n-1}{j}^{-1} \sum_{1 \leq i_1 < \cdots < i_j \leq n-1} k_{i_1} \cdots k_{i_j}$$

for $j = 1, \ldots, n-1$ and $H_0 = 1$.

$$H_1 = \frac{1}{n-1} \sum_{1 \leq i \leq n-1} k_i$$

the mean curvature

$$H_{n-1} = \prod_{i=1}^{n-1} k_i$$

the Gauss curvature
Coefficients in $L_p$ Steiner formula

For a general convex body $K$, the $L_p$ Steiner coefficients are defined as

$$W_{m, k}^p(K) = \int_{\partial K} \left< x, N(x) \right>^{m-k+\frac{n(1-p)}{n+p}} H_{n-1}^{\frac{p}{n+p}} \sum_{i_1, \ldots, i_{n-1} \geq 0 \atop i_1+2i_2+\cdots+(n-1)i_{n-1}=m} c(n, p, i(m)) \prod_{j=1}^{n-1} \binom{n-1}{j}^{i_j} H_j^{i_j} \, d\mathcal{H}^{n-1},$$

where

$$c(n, p, i(m)) = \binom{n}{\frac{n}{n+p}} \binom{i_1 + \cdots + i_{n-1}}{i_1, i_2, \ldots, i_{n-1}},$$

such that the sequence $i(m) = \{i_j\}_{j=0}^{n-1}$ satisfies $i_1 + 2i_2 + \cdots + (n-1)i_{n-1} = m$. 
Coefficients in \( L_p \) Steiner formula

For a general convex body \( K \), the \( L_p \) Steiner coefficients are defined as

\[
\mathcal{W}^p_{m, k}(K) = \int_{\partial K} \langle x, N(x) \rangle^{m-k+\frac{n(1-p)}{n+p}} H_{n-1}^{n+p} \sum_{i_1, \ldots, i_{n-1} \geq 0 \atop i_1 + 2i_2 + \cdots + (n-1)i_{n-1} = m} c(n, p, i(m)) \prod_{j=1}^{n-1} \binom{n-1}{j}^i_j H_j^{i_j} d\mathcal{H}^{n-1},
\]

where

\[
c(n, p, i(m)) = \binom{n}{i_1 + \cdots + i_{n-1}} \left( \frac{n}{n+p} \right)^{i_1 + \cdots + i_{n-1}} \left( i_1, i_2, \ldots, i_{n-1} \right)
\]

such that the sequence \( i(m) = \{i_j\}_{j=0}^{n-1} \) satisfies \( i_1 + 2i_2 + \cdots + (n-1)i_{n-1} = m \).

In analogy to the classical Steiner formula, for a general convex body \( K \) in \( \mathbb{R}^n \) the \( L_p \)-Steiner quermassintegrals are defined as

\[
\mathcal{V}^p_k(K) = \sum_{m=0}^{k} \binom{n(1-p)}{n+p} \binom{k}{m} \mathcal{W}^p_{m, k}(K).
\]
Special cases: $p = 0$

If $K$ is $C^2_+$, then

$$\mathcal{V}_k^0(K) = n \binom{n}{k} W_k(K) = \binom{n}{k} \int_{\partial K} H_{k-1} d\mathcal{H}^{n-1}.$$ 

In particular,

$$\mathcal{V}_0^0(K) = n \text{Vol}_n(K) = as_0(K).$$

**Corollary (Classical Steiner formula, $p = 0$)**

*Let $K$ be a convex body in $\mathbb{R}^n$ that is $C^2_+$. Then*

$$as_0(K + tB_2^n) = \sum_{i=0}^{n} \binom{n}{i} W_i(K) t^i.$$
Corollary \((p = \frac{n(1-l)}{l}, \ l \in \mathbb{N})\)

Let \(K\) be a convex body in \(\mathbb{R}^n\) that is \(C^2_+\). Then

\[
\text{as}_{\frac{n(1-l)}{l}}(K + tB^n_2) = \sum_{k=0}^{n(2l-1)} \mathcal{V}_k^p(K) \ t^k.
\]

Note that

- \(\mathcal{V}_0^p(K) = \text{as}_p(K)\) (true for arbitrary \(p\)):
Special cases: \( \frac{n}{n+p} = l \in \mathbb{N} \)

**Corollary** \((p = \frac{n(1-l)}{l}, \ l \in \mathbb{N})\)

Let \( K \) be a convex body in \( \mathbb{R}^n \) that is \( C^2_+ \). Then

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\text{as}_{\frac{n(1-l)}{l}}(K + tB_n^2) = \sum_{k=0}^{n(2l-1)} \mathcal{V}_k^p(K) t^k.
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Note that

- \( \mathcal{V}_0^p(K) = \text{as}_p(K) \) (true for arbitrary \( p \)):

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\mathcal{V}_k^p(K) = \sum_{m=0}^{k} \binom{n(1-p)}{n+p} \mathcal{W}_{m,k}^p(K)
\]
Corollary \((p = \frac{n(1-l)}{l}, \ l \in \mathbb{N})\)

Let \(K\) be a convex body in \(\mathbb{R}^n\) that is \(C^2_+\). Then

\[
\text{as}_{\frac{n(1-l)}{l}} (K + tB^n_2) = \sum_{k=0}^{\frac{n(2l-1)}{l}} \mathcal{V}_k^p (K) \ t^k.
\]

Note that

- \(\mathcal{V}_0^p (K) = as_p(K)\) (true for arbitrary \(p\)):

\[
k = 0 : \quad \mathcal{V}_0^p (K) = \sum_{m=0}^{0} \left( \frac{n(1-p)}{n+p} \right) \mathcal{W}_{m,0}^p (K)
\]
Corollary \((p = \frac{n(1-l)}{l}, \ l \in \mathbb{N})\)

Let \(K\) be a convex body in \(\mathbb{R}^n\) that is \(C^2_+\). Then

\[
\text{as}_{\frac{n(1-l)}{l}} (K + tB_2^n) = \sum_{k=0}^{\frac{n(2l-1)}{l}} \mathcal{V}_k^p(K) \ t^k.
\]

Note that

- \(\mathcal{V}_0^p(K) = \text{as}_p(K)\) (true for arbitrary \(p\)):

\[
k = 0 : \quad \mathcal{V}_0^p(K) \equiv 0 \quad \left(\frac{n(1-p)}{n+p}\right) \mathcal{W}_{0,0}^p(K)
\]
Special cases: \( \frac{n}{n + p} = l \in \mathbb{N} \)

**Corollary** \((p = \frac{n(1-l)}{l}, \ l \in \mathbb{N})\)

*Let \( K \) be a convex body in \( \mathbb{R}^n \) that is \( C^2_+ \). Then*

\[
\text{as}_{\frac{n(1-l)}{l}}(K + tB^n_2) = \sum_{k=0}^{n(2l-1)} V^p_k(K) t^k.
\]

Note that

- \( V^p_0(K) = \text{as}_p(K) \) (true for arbitrary \( p \)):

\[
k = 0 : \quad V^p_0(K) = \mathcal{W}^p_{0,0}(K) =
\]

\[
\mathcal{W}^p_{m,k}(K) = \int_{\partial K} \left\langle x, N(x) \right\rangle^{m-k+\frac{n(1-p)}{n+p}} H_{n-1}^{\frac{p}{n+p}} \sum_{i_1, \ldots, i_{n-1} \geq 0} c(n, p, i(m)) \prod_{j=1}^{n-1} \left( \frac{n-1}{j} \right)^{i_j} H_j^{i_j} d\mathcal{H}^{n-1}
\]

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Special cases: \( \frac{n}{n+p} = 1 \in \mathbb{N} \)

**Corollary (p = \( \frac{n(1-l)}{l} \), l \in \mathbb{N})**

Let \( K \) be a convex body in \( \mathbb{R}^n \) that is \( C^2_+ \). Then

\[
\text{as}_{\frac{n(1-l)}{l}}(K + tB^n_2) = \sum_{k=0}^{n(2l-1)} \mathcal{V}_k^p(K) t^k.
\]

Note that

- \( \mathcal{V}_0^p(K) = \text{as}_p(K) \) (true for arbitrary \( p \):

\[
k = 0: \quad \mathcal{V}_0^p(K) = \mathcal{W}_{0,0}^p(K) = \int_{\partial K} \frac{H^{\frac{p}{n+p}}}{\langle x, N(x) \rangle^{\frac{n(p-1)}{n+p}}} d\mathcal{H}^{n-1} = \text{as}_p(K)
\]

\[
\mathcal{W}_{m,k}^p(K) = \int_{\partial K} \langle x, N(x) \rangle^{m-k+\frac{n(1-p)}{n+p}} \frac{H^{\frac{p}{n+p}}}{H^{\frac{n}{n-1}}} \sum_{i_1, \ldots, i_{n-1} \geq 0 \atop i_1 + 2i_2 + \cdots + (n-1)i_{n-1} = m} c(n, p, i(m)) \prod_{j=1}^{n-1} \binom{n-1}{j}^{i_j} H_j^{i_j} d\mathcal{H}^{n-1}
\]
Special cases: \( \frac{n}{n+p} = l \in \mathbb{N} \)

**Corollary** \((p = \frac{n(1-l)}{l}, l \in \mathbb{N})\)

Let \( K \) be a convex body in \( \mathbb{R}^n \) that is \( C^2_+ \). Then

\[
\text{as}_{\frac{n(1-l)}{l}} (K + tB^n_2) = \sum_{k=0}^{n(2l-1)} V^p_k (K) t^k.
\]

Note that

- \( V^p_0 (K) = as_p (K) \) (true for arbitrary \( p \));
- \( V^p_{\frac{n(2l-1)}{l}} (K) = as_p (B^n_2) \).
Special cases: $p = \pm \infty$

Recall that dual quermassintegrals of order $i$ are

$$\tilde{W}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K(u)^{n-i} d\sigma(u).$$

Then

$$V_{\pm \infty}^k(K) = \left(\begin{array}{c} -n \\ k \end{array}\right) \tilde{W}_k(K^\circ) = (-1)^k \binom{n + k - 1}{k} \tilde{W}_k(K^\circ).$$

Corollary (Steiner formula for Minkowski outer parallel body of the dual theory, $p = \pm \infty$)

Let $K$ be a convex body in $\mathbb{R}^n$ that is $C^2$. Let $t \in \mathbb{R}$ be such that $0 \leq t < \min_{u \in S^{n-1}} h_K(u)$. Then

$$as_{\pm \infty}(K + tB_2^n) = n \text{Vol}_n((K + tB_2^n)^\circ) = n \sum_{i=0}^{\infty} \left(\begin{array}{c} -n \\ i \end{array}\right) \tilde{W}_{-i}(K^\circ) t^i.$$
According to the $L_p$ Steiner formula (with $K = B_2^n$) we get

$$\text{as}_p(B_2^n + tB_2^n) = \sum_{k=0}^{\infty} \mathcal{V}_k^p(B_2^n) t^k.$$ 

On the other hand,

$$\text{as}_p((1 + t)B_2^n) = \left[ \text{as}_p(\lambda K) = \lambda^{n - p \over n + p} \text{as}_p(K) \right] = (1 + t)^{n - p \over n + p} \text{as}_p(B_2^n)$$

$$= \left[ \text{as}_p(B_2^n) = \text{Vol}_{n-1}(\partial B_2^n) \right] = (1 + t)^{n - p \over n + p} \text{Vol}_{n-1}(\partial B_2^n)$$

$$= \text{Vol}_{n-1}(\partial B_2^n) \sum_{k=0}^{\infty} \binom{n - p \over n + p}{k} t^k,$$

and by definition

$$\mathcal{V}_k^p(B_2^n) = \text{Vol}_{n-1}(\partial B_2^n) \sum_{m=0}^{k} \binom{n(1-p) \over n+p}{k-m} \sum_{i_1, \ldots, i_{n-1} \geq 0 \atop i_1 + 2i_2 + \cdots + (n-1)i_{n-1} = m} c(n, p, i(m)) \prod_{j=1}^{n-1} \binom{n - 1 \over j}{i_j}.$$
Euclidean ball

According to the $L_p$ Steiner formula (with $K = B_2^n$) we get

$$as_p(B_2^n + tB_2^n) = \sum_{k=0}^{\infty} \mathcal{V}_k^p(B_2^n) t^k.$$ 

On the other hand,

$$as_p((1 + t)B_2^n) = \left[ as_p(\lambda K) = \lambda^{n-\frac{p}{n+p}} as_p(K) \right] = (1 + t)^{n-\frac{p}{n+p}} as_p(B_2^n)$$

$$= [as_p(B_2^n) = Vol_{n-1}(\partial B_2^n)] = (1 + t)^{n-\frac{p}{n+p}} Vol_{n-1}(\partial B_2^n)$$

$$= Vol_{n-1}(\partial B_2^n) \sum_{k=0}^{\infty} \binom{n-\frac{p}{n+p}}{k} t^k,$$

and by definition

$$\mathcal{V}_k^p(B_2^n) = Vol_{n-1}(\partial B_2^n) \sum_{m=0}^{k} \binom{n-\frac{p}{n+p}}{k-m} \sum_{i_1, \ldots, i_{n-1} \geq 0} c(n, p, i(m)) \prod_{j=1}^{n-1} \binom{n-1}{j} ^{i_j}.$$ 

$$C(n,p,k)$$
Thus,
\[
\left( n^{\frac{n-p}{n+p}} \right)^k \text{Vol}_{n-1}(\partial B_2^n) = \mathcal{V}_k^p(B_2^n) = C(n, p, k)\text{Vol}_{n-1}(\partial B_2^n)
\]
where
\[
C(n, p, k) = \sum_{m=0}^{k} \left( \frac{n(1-p)}{n+p} \right)^k \sum_{i_1, \ldots, i_{n-1} \geq 0} c(n, p, i(m)) \prod_{j=1}^{n-1} \binom{n-1}{j}^{i_j}.
\]

**Corollary**

Let \( p \in \mathbb{R}, p \neq -n. \) Then
\[
\left( n^{\frac{n-p}{n+p}} \right)^k = \sum_{m=0}^{k} \left( \frac{n(1-p)}{n+p} \right)^k \sum_{i_1, \ldots, i_{n-1} \geq 0} c(n, p, i(m)) \prod_{j=1}^{n-1} \binom{n-1}{j}^{i_j}.
\]
Homogeneity

**Theorem**

Let $K$ be a convex body in $\mathbb{R}^n$. Then for all $p \in \mathbb{R}$, $p \neq -n$, and for all $k \in \mathbb{N}$, $\mathcal{V}_p^p(K)$ (and $\mathcal{W}_p^p,m,k(K)$) are homogeneous of degree $n \frac{n-p}{n+p} - k$.

**Remark.** When $K$ is $C^2_+$, we have

$$as_p(\lambda K + tB^n_2) = as_p \left( \lambda \left( K + \frac{t}{\lambda} B^n_2 \right) \right) = \lambda^{n \frac{n-p}{n+p}} as_p \left( K + \frac{t}{\lambda} B^n_2 \right) = \sum_{k=0}^{\infty} \lambda^{n \frac{n-p}{n+p} - k} \mathcal{V}_p^p(K) t^k.$$

On the other hand, we get

$$as_p(\lambda K + tB) = \sum_{k=0}^{\infty} \mathcal{V}_p^p(\lambda K) t^k.$$

Thus,

$$\mathcal{V}_p^p(\lambda K) = \lambda^{n \frac{n-p}{n+p} - k} \mathcal{V}_p^p(K).$$
Homogeneity

Proof.

Proposition

Let $g : \partial K \to \mathbb{R}$ an integrable function, and $T : \mathbb{R}^n \to \mathbb{R}^n$ an invertible, linear map. Then

$$
\int_{\partial K} g(x) \, d\mathcal{H}^{n-1}(x) = |\det(T)|^{-1} \int_{\partial T(K)} \| T^{-1}(N_K(T^{-1}(y))) \|^{-1} g(T^{-1}(y)) \, d\mathcal{H}^{n-1}(x)
$$

and

$$
\langle T^{-1}(y), N_K(T^{-1}y) \rangle = \langle y, N_{T(K)}(y) \rangle \| T^{-1}(N_K(T^{-1}(y))) \|
$$

for all $y \in \partial T(K)$.

Applying this proposition with $T = \lambda Id$, and using that for any $y \in \partial(\lambda K)$,

$$
H_j(y) = \frac{H_j(T^{-1}y)}{\lambda^j},
$$
Proof.

\[ W^p_{m,k}(K) = \lambda^{k-n} \int_{\partial(\lambda K)} \langle y, N_{\lambda K}(y) \rangle^{m-k+n/(n+p)} H_{n-1}^{n+p}(y) \]

\[ \sum_{i_1, \ldots, i_{n-1} \geq 0 \atop i_1 + 2i_2 + \cdots + (n-1)i_{n-1} = m} c(n, p, i(m)) \prod_{j=1}^{n-1} \binom{n-1}{j}^i H_j^i(y) \, d\mathcal{H}^{n-1}(y) \]

\[ = \lambda^{k-n} W^p_{m,k}(\lambda K). \]

Therefore,

\[ V^p_k(\lambda K) = \sum_{m=0}^{k} \binom{n(1-p)}{n+p} \lambda^{k-n} W^p_{m,k}(\lambda K) = \sum_{m=0}^{k} \binom{n(1-p)}{n+p} \lambda^{k-n} W^p_{m,k}(K) = \lambda^{k-n} V^p_k(K). \]
Invariance

Theorem

Let $K$ be a convex body in $\mathbb{R}^n$. Then for all $p \in \mathbb{R}$, $p \neq -n$, and for all $k \in \mathbb{N}$, $\mathcal{V}_k^p(K)$ (and $\mathcal{W}_{m,k}^p(K)$) are invariant under rotations and reflections. When $p = 1$, $\mathcal{V}_k^1(K)$ (and $\mathcal{W}_{k,k}(K)$) are also invariant under translations.

Proof.

If $T$ is a rotation or a reflection, then

- $|\det T| = 1$;
- $\|T^{-1}(N_K(T^{-1}(y)))\| = \|N_K(T^{-1}(y))\| = 1$;
- $\{H_j(y) : y \in \partial T(K)\} = \{H_j(x) : x \in \partial K\}$ for all $1 \leq j \leq n - 1$.

Applying the previous proposition, we get

$$\mathcal{W}_{m,k}^p(K) = \mathcal{W}_{m,k}^p(T(K)).$$

Thus,

$$\mathcal{V}_k^p(K) = \sum_{m=0}^{k} \left( \frac{n(1-p)}{n+p} \right)^{k-m} \mathcal{W}_{m,k}^p(K) = \sum_{m=0}^{k} \left( \frac{n(1-p)}{n+p} \right)^{k-m} \mathcal{W}_{m,k}(T(K)) = \mathcal{V}_m^p(T(K)).$$
Let $K$ be a convex body in $\mathbb{R}^n$. Then for $p \geq 0$ and $p < -n$, and for all $k \in \mathbb{N}$, $m \in \mathbb{N}$, $\mathcal{W}^p_{m,k}$ are valuations. Moreover, $\mathcal{V}^p_k$ are also valuations.

**Remark.** (when $K$ is $C^2_\pm$)

Let $K$ and $L$ be convex bodies in $\mathbb{R}^n$ such that $K \cup L$ is a convex body. We need to show that
\[ \mathcal{V}^p_k(K \cup L) + \mathcal{V}^p_k(K \cap L) = \mathcal{V}^p_k(K) + \mathcal{V}^p_k(L). \] (1)

Using the fact that $L_p$ affine surface area is a valuation,
\[ as_p((K + tB^n_2) \cup (L + tB^n_2)) + as_p((K + tB^n_2) \cap (L + tB^n_2)) = as_p(K + tB^n_2) + as_p(L + tB^n_2), \]
we get
\[ as_p(K \cup L + tB^n_2) + as_p(K \cap L + tB^n_2) = as_p(K + tB^n_2) + as_p(L + tB^n_2). \]

After applying $L_p$ Steiner formula for each term and collecting coefficients, we get (1).
Valuation: $p = 1$

**Theorem**

Let $K$ be a convex body in $\mathbb{R}^n$. Then for all $k \in \mathbb{N}$, $m \in \mathbb{N}$, $V_k^1 = \mathcal{V}_{k, k}$ are valuations.

We want to show that for convex bodies $K$ and $L$ in $\mathbb{R}^n$ such that $K \cup L$ is a convex body,

$$\mathcal{V}_{k, k}(K \cup L) + \mathcal{V}_{k, k}(K \cap L) = \mathcal{V}_{k, k}(K) + \mathcal{V}_{k, k}(L).$$

Note that

$$\mathcal{V}_{k, k}(K) = \int_{\partial K} H_{n-1}^{1}(x) \sum_{i_1, \ldots, i_{n-1} \geq 0, i_1 + 2i_2 + \cdots + (n-1)i_{n-1} = k} c(n, 1, i(k)) \prod_{j=1}^{n-1} \left( \begin{array}{c} n - 1 \\ j \end{array} \right)^{ij} H_j^i(x) \, d\mathcal{H}^{n-1}(x)$$

is a sum of (up to some constants) integrals of the form

$$\int_{\partial K} H_{n-1}^{1}(x) \prod_{i=1}^{j} k_{ij}^{\alpha_j}(x) \, d\mathcal{H}^{n-1}(x).$$
Valuation: $p = 1$

**Theorem**

Let $K$ be a convex body in $\mathbb{R}^n$. For all $1 \leq i_1, \ldots, i_j \leq n - 1$ and $\alpha_1, \alpha_2, \ldots, \alpha_j \geq 0$,

$$
\int_{\partial K} H_{n-1}^{\frac{1}{n+1}}(x) \prod_{i=1}^{j} k_{ij}^{\alpha_j}(x) \, d\mathcal{H}^{n-1}(x)
$$

is a valuation.

Let $K$ and $L$ be convex bodies in $\mathbb{R}^n$ such that $K \cup L$ is a convex body. We decompose a body using a strategy that was introduced by Schütt:

$$
\begin{align*}
\partial(K \cup L) &= \{\partial K \cap \partial L\} \cup \{\partial K \cap L^c\} \cup \{K^c \cap \partial L\} \\
\partial(K \cap L) &= \{\partial K \cap \partial L\} \cup \{\partial K \cap \text{int}(L)\} \cup \{\text{int}(K) \cap \partial L\} \\
\partial K &= \{\partial K \cap \partial L\} \cup \{\partial K \cap L^c\} \cup \{\partial K \cap \text{int}(L)\} \\
\partial L &= \{\partial K \cap \partial L\} \cup \{\partial L \cap K^c\} \cup \{\partial L \cap \text{int}(K)\}.
\end{align*}
$$
Let $K$ and $L$ be convex bodies in $\mathbb{R}^n$ such that $K \cup L$ is a convex body. We decompose

\[
\partial(K \cup L) = \{\partial K \cap \partial L\} \cup \{\partial K \cap L^c\} \cup \{K^c \cap \partial L\}
\]
\[
\partial(K \cap L) = \{\partial K \cap \partial L\} \cup \{\partial K \cap \text{int}(L)\} \cup \{\text{int}(K) \cap \partial L\}
\]
\[
\partial K = \{\partial K \cap \partial L\} \cup \{\partial K \cap L^c\} \cup \{\partial K \cap \text{int}(L)\}
\]
\[
\partial L = \{\partial K \cap \partial L\} \cup \{\partial L \cap K^c\} \cup \{\partial L \cap \text{int}(K)\}.
\]

We also note that for all $x \in \partial K \cap \partial L$, and for all $\alpha \geq 0$ where the principal curvatures $k_j(\partial(K \cup L), x), k_j(\partial(K \cap L), x), k_j(K, x)$ and $k_j(L, x)$ exist for all $1 \leq i_1, \ldots, i_j \leq n - 1$,

\[
H_{n-1}(\partial(K \cup L), x) \frac{1}{n+1} k_j(\partial(K \cup L), x)^\alpha = \min \{H_{n-1}(K, x) \frac{1}{n+1} k_j(K, x)^\alpha, H_{n-1}(L, x) \frac{1}{n+1} k_j(L, x)^\alpha\}
\]

and

\[
H_{n-1}(\partial(K \cap L), x) \frac{1}{n+1} k_j(\partial(K \cap L), x)^\alpha = \max \{H_{n-1}(K, x) \frac{1}{n+1} k_j(K, x)^\alpha, H_{n-1}(L, x) \frac{1}{n+1} k_j(L, x)^\alpha\}.
\]
Continuity: general case

- $p \geq 1$: $a_{sp}(K)$ is upper semi continuous ($Ludwig$, $Lutwak$);
- $0 \leq p < 1$: $a_{sp}(K)$ is upper semi continuous ($Ludwig$, $Hug$);
- $-n < p < 0$: $a_{sp}(K)$ is lower semi continuous ($Ludwig$).

**Question:** are $L_p$-Steiner quermassintegrals $\mathcal{V}_k^p$ and $L_p$ Steiner coefficients $\mathcal{W}_{m,k}^p$ (upper or lower semi) continuous?
Continuity: general case

- **$p \geq 1$:** $a_s(K)$ is upper semi continuous (Ludwig, Lutwak);
- **$0 \leq p < 1$:** $a_s(K)$ is upper semi continuous (Ludwig, Hug);
- **$-n < p < 0$:** $a_s(K)$ is lower semi continuous (Ludwig).

**Question:** are $L_p$-Steiner quermassintegrals $\gamma^p_k$ and $L_p$ Steiner coefficients $\gamma^p_{m,k}$ (upper or lower semi) continuous?

**Proposition**

Let $p \neq -n$ and let $k \geq 1$. Then $\gamma^p_k$ are in general neither lower semi continuous nor upper semi continuous.
Continuity: $p = 1$

**Proposition**

Let $k \geq 1$. Then $V_k^1 = W_k$, $k$ are neither lower semi continuous nor upper semi continuous.
Continuity: $p = 1$

**Proposition**

Let $k \geq 1$. Then $V_k^1 = \mathcal{W}_k$, $k$ are neither lower semi continuous nor upper semi continuous.

Consider convex bodies $K_l, l \in \mathbb{N}$:

$$K_l = \left(1 - \frac{1}{l}\right) B_n^\infty + \frac{1}{l} B_2^n,$$

where $B_n^\infty = \{x \in \mathbb{R}^n : \|x\|_\infty \leq 1\}$ and $B_2^n$ is the Euclidean ball. Then $K_l \rightarrow B_n^\infty$ in Hausdorff metric.
Continuity: $p = 1$

### Proposition

Let $k \geq 1$. Then $\mathcal{V}^1_k = \mathcal{W}_{k,k}$ are neither lower semi continuous nor upper semi continuous.

Consider convex bodies $K_l$, $l \in \mathbb{N}$:

$$K_l = \left(1 - \frac{1}{l}\right) B^n_\infty + \frac{1}{l} B^n_2,$$

where $B^n_\infty = \{x \in \mathbb{R}^n : \|x\|_\infty \leq 1\}$ and $B^n_2$ is the Euclidean ball. Then $K_l \to B^n_\infty$ in Hausdorff metric.

Since $\mathcal{W}_{k,k}(B^n_\infty) = 0$,

$$\mathcal{W}_{k,k}(K_l) = \int_{\partial K_l} H_{n-1}^{\frac{1}{n+1}} \sum_{i_1, \ldots, i_{n-1} \geq 0 \atop i_1 + 2i_2 + \cdots + (n-1)i_{n-1} = k} c(n, 1, i(k)) \prod_{j=1}^{n-1} \binom{n-1}{j}^{i_j} H_j^{i_j}(x) \, d\mathcal{H}^{n-1}(x).$$
Continuity: $p = 1$

**Proposition**

Let $k \geq 1$. Then $\mathcal{V}_k^1 = \mathcal{W}_{k,k}$ are neither lower semi continuous nor upper semi continuous.

Consider convex bodies $K_l$, $l \in \mathbb{N}$:

$$K_l = \left(1 - \frac{1}{l}\right) B^n_\infty + \frac{1}{l} B^n_2,$$

where $B^n_\infty = \{x \in \mathbb{R}^n : \|x\|_\infty \leq 1\}$ and $B^n_2$ is the Euclidean ball. Then $K_l \to B^n_\infty$ in Hausdorff metric.

We denote by $B^n_2(x_0, r)$ the Euclidean ball with center at $x_0$ and radius $r$.

Since $\mathcal{W}_{k,k}(B^n_\infty) = 0$,

$$\mathcal{W}_{k,k}(K_l) = \int_{\partial B^n_2(0, \frac{1}{l})} H^{\frac{1}{n+1}} \sum_{i_1, \ldots, i_{n-1} \geq 0} c(n, 1, i(k)) \prod_{j=1}^{n-1} \binom{n-1}{j} \int_{\mathbb{R}^n} H^i_j(x) \, d\mathcal{H}^{n-1}(x)$$

$$= \mathcal{W}_{k,k} \left( B^n_2 \left(0, \frac{1}{l}\right) \right) = l^{k-n} \frac{n-1}{n+1} \text{Vol}_{n-1} (\partial B^n_2) C(n, 1, k).$$
Continuity: \( p = 1 \)

Therefore,
\[
\mathcal{W}_{k, k}(K_l) = l^{k-n} n^{-1} \frac{n+1}{n+1} \text{Vol}_{n-1} \left( \partial B_2^n \right) C(n, 1, k),
\]
where
\[
C(n, 1, k) = \binom{n}{k} \frac{n+1}{n+1}.
\]

- If \( k \geq n \) and
  - if \( k - n + 1 \) is even, then \( C(n, 1, k) > 0 \) which implies that
    \[
    \mathcal{W}_{k, k}(K_l) = l^{k-n} n^{-1} \frac{n+1}{n+1} \text{Vol}_{n-1} \left( \partial B_2^n \right) C(n, 1, k) \to \infty \text{ as } l \to \infty.
    \]
    Thus, \( \mathcal{W}_{k, k} \) is not upper semi continuous (since \( \mathcal{W}_{k, k}(B_2^n) = 0 \)).

  If we take a sequence of polytopes \( P_l \) that converge to \( B_2^n \) in Hausdorff metric, then
  \( \mathcal{W}_{k, k}(P_l) = 0 \), but \( \mathcal{W}_{k, k}(B_2^n) = \text{Vol}_{n-1} \left( \partial B_2^n \right) C(n, 1, k) > 0 \), so \( \mathcal{W}_{k, k} \) is not lower semi continuous.

  - if \( k - n + 1 \) is odd, then then \( C(n, 1, k) < 0 \) which similarly implies that \( \mathcal{W}_{k, k} \) is not upper or lower semi continuous.

- if \( k \leq n - 1 \), \( \mathcal{W}_{k, k} \) is not lower semi continuous since \( C(n, 1, k) > 0 \) and not upper semi continuous.
For all $p \neq -n$ and all $s \in \mathbb{R}$, the $s$-th mixed $L_p$ affine surface area of $K$ (Lutwak (1987) for $p \geq 1$ and all $s \in \mathbb{R}$; Werner, Ye (2010) for all $p \neq -n$ and all $s$) is defined as

$$\alpha_{p, s}(K) = \int_{\partial K} H_{n-1}(x)^{\frac{s+p}{n+p}} \langle x, N(x) \rangle^{(1-p)\frac{n-s}{n+p}} d\mathcal{H}^{n-1}(x).$$
Continuity: mixed affine surface areas

For all \( p \neq -n \) and all \( s \in \mathbb{R} \), the \( s \)-th mixed \( L_p \) affine surface area of \( K \) (Lutwak (1987) for \( p \geq 1 \) and all \( s \in \mathbb{R} \); Werner, Ye (2010) for all \( p \neq -n \) and all \( s \)) is defined as

\[
as_{p,s}(K) = \int_{\partial K} H_{n-1}(x)^{\frac{s+p}{n+p}} \langle x, N(x) \rangle^{(1-p)\frac{n-s}{n+p}} d\mathcal{H}^{n-1}(x).
\]

Special cases of the \( L_p \) Steiner coefficients:

- \( k = m = l(n-1), l \in \mathbb{N} \) and \( p = 1 \)

\[
\mathcal{V}^1_{l(n-1)}(K) = \mathcal{W}_{l(n-1), l(n-1)}(K) = \left( \frac{n}{n+1} \right) as_{1,l(n+1)}(K).
\]

- \( m = 0 \)

\[
\mathcal{W}^p_{0,k}(K) = \int_{\partial K} \langle x, N(x) \rangle^{-k+\frac{n(1-p)}{n+p}} H_{n-1}^{\frac{p}{n+p}} d\mathcal{H}^{n-1}(x) = as_{p+\frac{k}{n}(n+p), -k}(K).
\]
For all $p \neq -n$ and all $s \in \mathbb{R}$, the $s$-th mixed $L_p$ affine surface area of $K$ (Lutwak (1987) for $p \geq 1$ and all $s \in \mathbb{R}$; Werner, Ye (2010) for all $p \neq -n$ and all $s$) is defined as

$$as_{p,s}(K) = \int_{\partial K} H_{n-1}(x)^{\frac{s+p}{n+p}} \langle x, N(x) \rangle^{(1-p)\frac{n-s}{n+p}} d\mathcal{H}^{n-1}(x).$$

Special cases of the $L_p$ Steiner coefficients:

- $k = m = l(n-1)$, $l \in \mathbb{N}$ and $p = 1$

$$\mathcal{V}^1_{l(n-1)}(K) = \mathcal{W}_{l(n-1), l(n-1)}(K) = \left(\frac{n}{n+1}\right) as_{1, l(n+1)}(K).$$

- $m = 0$

$$\mathcal{W}^p_{0, k}(K) = \int_{\partial K} \langle x, N(x) \rangle^{-k+\frac{n(1-p)}{n+p}} H_{n-1}^{\frac{p}{n+p}} d\mathcal{H}^{n-1}(x) = as_{p+\frac{k}{n} (n+p), -k}(K).$$

**Theorem**

For $p \geq 0$, $\mathcal{W}^p_{0, k} = as_{p+\frac{k}{n} (n+p), -k}$ are upper semi continuous $n\frac{n-p}{n+p} - k$ homogeneous valuations that are invariant under rotation and reflections.
Thank you!