L_p Steiner formula and its coefficients

Kateryna Tatarko
(joint work with Elisabeth Werner)

University of Waterloo

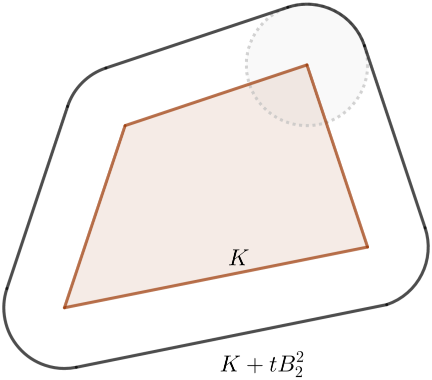
Phenomena in High Dimension

Classical Steiner formula

For a convex body K and $t \geq 0$, the **classical Steiner formula** is

$$\mathrm{Vol}_n(K + tB_2^n) = \sum_{i=0}^{n} \binom{n}{i} W_i(K) t^i = \sum_{i=0}^{n} \mathrm{Vol}_{n-i}(B_2^{n-i})\ V_i(K) t^{n-i},$$

where B_2^n is a Euclidean unit ball in $\mathbb{R}^n$.



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The coefficients $W_i(K)$ and $V_i(K)$ are called quermassintegrals and intrinsic volumes, respectively.

In particular,

$W_0(K) = \mathrm{Vol}_n(K)$	the volume
$W_1(K) = \frac{1}{n}\mathrm{Vol}_{n-1}(\partial K)$	the surface area
$W_n(K) = \mathrm{Vol}_n(B_2^n)$	the volume of the unit ball

Steiner formula of the dual Brunn Minkowski theory

The radial function of a star body K is defined by

$$\rho_K(u) = \max\{\lambda \geq 0 \,|\, \lambda u \in K\} \quad \text{for any } u \in S^{n-1}.$$

For star bodies K and L, the radial sum $\alpha K \widetilde{+} \beta L$ is the star body with the radial function

$$\rho_{\alpha K \widetilde{+} \beta L}(u) = \alpha \rho_K(u) + \beta \rho_L(u).$$

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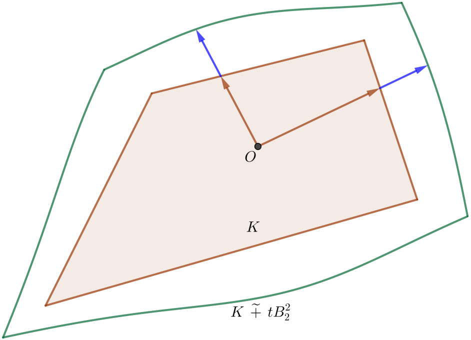

For a star body K and $t \geq 0$, we have that

$$\mathrm{Vol}_n(K \,\widetilde{+}\, tB_2^n) = \sum_{i=0}^{n} \binom{n}{i} \widetilde{W}_i(K) t^i,$$

where $\widetilde{+}$ is a radial addition. The coefficients $\widetilde{W}_i(K)$ are called dual quermassintegrals that were introduced by Lutwak.

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where $\widetilde{+}$ is a radial addition. The coefficients $\widetilde{W_i}(K)$ are called dual quermassintegrals that were introduced by Lutwak.

Dual quermassintegrals of order i can be written as

$$\widetilde{W_i}(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K(u)^{n-i} d\sigma(u).$$

L_p affine surface area

For real $p \neq -n$, the L_p affine surface area is defined as

$$as_p(K) = \int_{\partial K} \frac{H_{n-1}(x)^{\frac{p}{n+p}}}{\langle x, N(x)\rangle^{\frac{n(p-1)}{n+p}}} d\mathcal{H}^{n-1}(x),$$

where $H_{n-1}(x)$ is the Gauss curvature of K at $x \in \partial K$ and $N(x)$ is the outer normal vector at x.

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where $H_{n-1}(x)$ is the Gauss curvature of K at $x \in \partial K$ and $N(x)$ is the outer normal vector at x.

When $p = 1$, we recover the classical affine surface area $as_1(K)$:

$$as_1(K) = \int\limits_{\partial K} H_{n-1}(x)^{\frac{1}{n+1}} \, d\mathcal{H}^{n-1}(x).$$

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For $p = 0$,

$$as_0(K) = \int\limits_{\partial K} \langle x, N(x)\rangle \, d\mathcal{H}^{n-1}(x) \;=\; n \,\mathrm{Vol}_n(K).$$

If the boundary of K is sufficiently smooth, then

$$as_p(K) = \int_{S^{n-1}} f_K(u)^{\frac{n}{n+p}} \, h_K(u)^{\frac{n(1-p)}{n+p}} \, d\sigma(u),$$

where $f_K(u)$ is the curvature function, i.e. $f_K(u)$ is the reciprocal of $H_{n-1}(x)$ at $x \in \partial K$ that has u as outer normal, and $h_K(u)$ is the support function of K.

L_p affine surface area over the sphere

If the boundary of K is sufficiently smooth, then

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For $p = \pm\infty$

$$as_{\pm\infty}(K) = \int_{S^{n-1}} \frac{1}{h_K(u)^n} d\sigma(u) \; = \; n \, \mathrm{Vol}_n(K^\circ),$$

where $K^\circ = \{y \in \mathbb{R}^n, \langle x, y \rangle \leq 1, \forall x \in K\}$ is the polar body of K.

Theorem (T., Werner, 2019)

Let K be a convex body in $\mathbb{R}^n$ that is C^2_+. Let $t \in \mathbb{R}$ be such that $0 \leq t < \min_{u \in S^{n-1}} h_K(u)$. For all $p \in \mathbb{R}$, $p \neq -n$,

$$as_p(K + tB_2^n) = \sum_{k=0}^{\infty} \left[\sum_{m=0}^{k} \binom{\frac{n(1-p)}{n+p}}{k-m} \mathcal{W}^p_{m,k}(K) \right] t^k = \sum_{k=0}^{\infty} \mathcal{V}^p_k(K) t^k.$$

In particular,

$$as_1(K + tB_2^n) = \sum_{k=0}^{\infty} \mathcal{W}_{k,k}(K)\, t^k = \sum_{k=0}^{\infty} \mathcal{V}^1_k(K)\, t^k.$$

The coefficients $\mathcal{W}^p_{m,k}(K)$ and $\mathcal{V}^p_k(K)$ are called L_p *Steiner coefficients* and L_p-*Steiner quermassintegrals*.

The j-th normalized elementary symmetric functions of the principal curvatures are

$$H_j = \binom{n-1}{j}^{-1} \sum_{1 \le i_1 < \cdots < i_j \le n-1} k_{i_1} \cdots k_{i_j}$$

for $j = 1, \ldots, n-1$ and $H_0 = 1$.

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$$H_1 = \frac{1}{n-1} \sum_{1 \le i \le n-1} k_i \qquad \text{the mean curvature}$$

$$H_{n-1} = \prod_{i=1}^{n-1} k_i \qquad \text{the Gauss curvature}$$

Coefficients in L_p Steiner formula

For a general convex body K, the L_p Steiner coefficients are defined as

$$\mathcal{W}^p_{m,\,k}(K) =$$

$$\int_{\partial K} \langle x, N(x)\rangle^{m-k+\frac{n(1-p)}{n+p}} H_{n-1}^{\frac{p}{n+p}} \sum_{\substack{i_1,\dots,i_{n-1}\geq 0\\ i_1+2i_2+\dots+(n-1)i_{n-1}=m}} c(n,p,i(m)) \prod_{j=1}^{n-1} \binom{n-1}{j}^{i_j} H_j^{i_j}\, d\mathcal{H}^{n-1},$$

where

$$c(n,p,i(m)) = \binom{\frac{n}{n+p}}{i_1+\dots+i_{n-1}} \binom{i_1+\dots+i_{n-1}}{i_1,\ i_2,\ \dots,\ i_{n-1}}$$

such that the sequence $i(m) = \{i_j\}_{j=0}^{n-1}$ satisfies $i_1 + 2i_2 + \ldots + (n-1)i_{n-1} = m$.

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In analogy to the classical Steiner formula, for a general convex body K in $\mathbb{R}^n$ the L_p-Steiner quermassintegrals are defined as

$$\mathcal{V}^p_k(K) = \sum_{m=0}^{k} \binom{\frac{n(1-p)}{n+p}}{k-m} \mathcal{W}^p_{m,k}(K).$$

Special cases: $p = 0$

If K is C^2_+, then

$$\mathcal{V}^0_k(K) = n\binom{n}{k} W_k(K) = \binom{n}{k} \int_{\partial K} H_{k-1} d\mathcal{H}^{n-1}.$$

In particular,

$$\mathcal{V}^0_0(K) = n\mathrm{Vol}_n(K) = as_0(K).$$

Corollary (Classical Steiner formula, $p = 0$)

Let K be a convex body in $\mathbb{R}^n$ that is C^2_+. Then

$$as_0(K + tB^n_2) = \sum_{i=0}^{n} \binom{n}{i} W_i(K)\ t^i.$$

Special cases: $\frac{n}{n+p} = l \in \mathbb{N}$

Corollary ($p = \frac{n(1-l)}{l}$, $l \in \mathbb{N}$)

Let K be a convex body in $\mathbb{R}^n$ that is C^2_+. Then

$$as_{\frac{n(1-l)}{l}}(K + tB_2^n) = \sum_{k=0}^{n(2l-1)} \mathcal{V}_k^p(K)\, t^k.$$

Note that

- $\mathcal{V}_0^p(K) = as_p(K)$ (true for arbitrary p):

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$$\mathcal{W}_{m,\,k}^p(K) =$$

$$\int_{\partial K} \langle x, N(x)\rangle^{m-k+\frac{n(1-p)}{n+p}} H_{n-1}^{\frac{p}{n+p}} \sum_{\substack{i_1,\ldots,i_{n-1}\geq 0 \\ i_1+2i_2+\cdots+(n-1)i_{n-1}=m}} c(n,p,i(m)) \prod_{j=1}^{n-1} \binom{n-1}{j}^{i_j} H_j^{i_j}\, d\mathcal{H}^{n-1}$$

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Note that

- $\mathcal{V}_0^p(K) = as_p(K)$ (true for arbitrary p):

$$k = 0: \quad \mathcal{V}_0^p(K) = \mathcal{W}_{0,0}^p(K) = \int_{\partial K} \frac{H_{n-1}^{\frac{p}{n+p}}}{\langle x, N(x)\rangle^{\frac{n(p-1)}{n+p}}}\, d\mathcal{H}^{n-1} = as_p(K)$$

$$\mathcal{W}_{m,\,k}^p(K) =$$

$$\int_{\partial K} \langle x, N(x)\rangle^{m-k+\frac{n(1-p)}{n+p}} H_{n-1}^{\frac{p}{n+p}} \sum_{\substack{i_1,\dots,i_{n-1}\geq 0\\ i_1+2i_2+\cdots+(n-1)i_{n-1}=m}} c(n,p,i(m)) \prod_{j=1}^{n-1} \binom{n-1}{j}^{i_j} H_j^{i_j}\, d\mathcal{H}^{n-1}$$

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Corollary ($p = \frac{n(1-l)}{l}$, $l \in \mathbb{N}$)

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$$as_{\frac{n(1-l)}{l}}(K + tB_2^n) = \sum_{k=0}^{n(2l-1)} \mathcal{V}_k^p(K)\, t^k.$$

Note that

- $\mathcal{V}_0^p(K) = as_p(K)$ (true for arbitrary p);
- $\mathcal{V}^p_{n(2l-1)}(K) = as_p(B_2^n)$.

Recall that dual quermassintegrals of order i are

$$\widetilde{W}_i(K) = \frac{1}{n} \int\limits_{S^{n-1}} \rho_K(u)^{n-i} d\sigma(u).$$

Then

$$\mathcal{V}_k^{\pm\infty}(K) = \binom{-n}{k} \widetilde{W}_{-k}(K^\circ) = (-1)^k \binom{n+k-1}{k} \widetilde{W}_{-k}(K^\circ)$$

Corollary (Steiner formula for Minkowski outer parallel body of the dual theory, $p = \pm\infty$)

Let K be a convex body in $\mathbb{R}^n$ that is C^2_+. Let $t \in \mathbb{R}$ be such that $0 \le t < \min_{u \in S^{n-1}} h_K(u)$. Then

$$as_{\pm\infty}(K + tB_2^n) \quad = \quad n\,\mathrm{Vol}_n\Big((K + tB_2^n)^\circ\Big) = n \sum_{i=0}^{\infty} \binom{-n}{i} \widetilde{W}_{-i}(K^\circ)\ t^i.$$

Euclidean ball

According to the L_p Steiner formula (with $K = B_2^n$) we get

$$as_p(B_2^n + tB_2^n) = \sum_{k=0}^{\infty} \mathcal{V}_k^p(B_2^n) t^k.$$

On the other hand,

$$\begin{aligned} as_p((1+t)B_2^n) &= \left[as_p(\lambda K) = \lambda^{n\frac{n-p}{n+p}} as_p(K)\right] = (1+t)^{n\frac{n-p}{n+p}} as_p(B_2^n) \\ &= [as_p(B_2^n) = \mathrm{Vol}_{n-1}(\partial B_2^n)] = (1+t)^{n\frac{n-p}{n+p}} \mathrm{Vol}_{n-1}(\partial B_2^n) \\ &= \mathrm{Vol}_{n-1}(\partial B_2^n) \sum_{k=0}^{\infty} \binom{n\frac{n-p}{n+p}}{k} t^k, \end{aligned}$$

and by definition

$$\mathcal{V}_k^p(B_2^n) = \mathrm{Vol}_{n-1}(\partial B_2^n) \sum_{m=0}^{k} \binom{\frac{n(1-p)}{n+p}}{k-m} \sum_{\substack{i_1,\dots,i_{n-1}\geq 0 \\ i_1+2i_2+\dots+(n-1)i_{n-1}=m}} c(n,p,i(m)) \prod_{j=1}^{n-1} \binom{n-1}{j}^{i_j}.$$

Euclidean ball

According to the L_p Steiner formula (with $K = B_2^n$) we get

$$as_p(B_2^n + tB_2^n) = \sum_{k=0}^{\infty} \mathcal{V}_k^p(B_2^n) t^k.$$

On the other hand,

$$\begin{aligned} as_p((1+t)B_2^n) &= \left[as_p(\lambda K) = \lambda^{n\frac{n-p}{n+p}} as_p(K) \right] = (1+t)^{n\frac{n-p}{n+p}} as_p(B_2^n) \\ &= \left[as_p(B_2^n) = \mathrm{Vol}_{n-1}(\partial B_2^n) \right] = (1+t)^{n\frac{n-p}{n+p}} \mathrm{Vol}_{n-1}(\partial B_2^n) \\ &= \mathrm{Vol}_{n-1}(\partial B_2^n) \sum_{k=0}^{\infty} \binom{n\frac{n-p}{n+p}}{k} t^k, \end{aligned}$$

and by definition

$$\mathcal{V}_k^p(B_2^n) = \mathrm{Vol}_{n-1}(\partial B_2^n) \underbrace{\sum_{m=0}^{k} \binom{\frac{n(1-p)}{n+p}}{k-m} \sum_{\substack{i_1,\dots,i_{n-1}\geq 0 \\ i_1+2i_2+\dots+(n-1)i_{n-1}=m}} c(n,p,i(m)) \prod_{j=1}^{n-1} \binom{n-1}{j}^{i_j}}_{C(n,p,k)}.$$

Euclidean ball

Thus,

$$\binom{n\frac{n-p}{n+p}}{k}\mathrm{Vol}_{n-1}(\partial B_2^n) = \mathcal{V}_k^p(B_2^n) = C(n,p,k)\mathrm{Vol}_{n-1}(\partial B_2^n)$$

where

$$C(n,p,k) = \sum_{m=0}^{k} \binom{\frac{n(1-p)}{n+p}}{k-m} \sum_{\substack{i_1,\dots,i_{n-1}\geq 0 \\ i_1+2i_2+\dots+(n-1)i_{n-1}=m}} c(n,p,i(m)) \prod_{j=1}^{n-1} \binom{n-1}{j}^{i_j}.$$

Corollary

Let $p \in \mathbb{R}$, $p \neq -n$. Then

$$\binom{n\frac{n-p}{n+p}}{k} = \sum_{m=0}^{k} \binom{\frac{n(1-p)}{n+p}}{k-m} \sum_{\substack{i_1,\dots,i_{n-1}\geq 0 \\ i_1+2i_2+\dots+(n-1)i_{n-1}=m}} c(n,p,i(m)) \prod_{j=1}^{n-1} \binom{n-1}{j}^{i_j}.$$

Theorem

Let K is a convex body in $\mathbb{R}^n$. Then for all $p \in \mathbb{R}$, $p \neq -n$, and for all $k \in \mathbb{N}$, $\mathcal{V}_k^p(K)$ (and $\mathcal{W}_{m,\,k}^p(K)$) are homogeneous of degree $n\frac{n-p}{n+p} - k$.

Remark. When K is C_+^2, we have

$$as_p(\lambda K + tB_2^n) = as_p\left(\lambda\left(K + \frac{t}{\lambda}B_2^n\right)\right) = \lambda^{n\frac{n-p}{n+p}} as_p\left(K + \frac{t}{\lambda}B_2^n\right) = \sum_{k=0}^{\infty} \lambda^{n\frac{n-p}{n+p}-k}\mathcal{V}_k^p(K)t^k.$$

On the other hand, we get

$$as_p(\lambda K + tB) = \sum_{k=0}^{\infty} \mathcal{V}_k^p(\lambda K)t^k.$$

Thus,

$$\mathcal{V}_k^p(\lambda K) = \lambda^{n\frac{n-p}{n+p}-k}\mathcal{V}_k^p(K).$$

Proof.

Proposition

Let $g : \partial K \to \mathbb{R}$ an integrable function, and $T : \mathbb{R}^n \to \mathbb{R}^n$ an invertible, linear map. Then

$$\int_{\partial K} g(x)\, d\mathcal{H}^{n-1}(x) = |\det(T)|^{-1} \int_{\partial T(K)} \|T^{-1t}(N_K(T^{-1}(y)))\|^{-1} g(T^{-1}(y))\, d\mathcal{H}^{n-1}(x)$$

and

$$\langle T^{-1}(y,)N_K(T^{-1}y)\rangle = \langle y, N_{T(K)}(y)\rangle \, \|T^{-1t}(N_K(T^{-1}(y)))\|$$

for all $y \in \partial T(K)$.

Applying this proposition with $T = \lambda Id$, and using that for any $y \in \partial(\lambda K)$,

$$H_j(y) = \frac{H_j(T^{-1}y)}{\lambda^j},$$

Proof.

$$\mathcal{W}^p_{m,\,k}(K) \qquad = \lambda^{k-n\frac{n-p}{n+p}} \int\limits_{\partial(\lambda K)} \langle y, N_{\lambda K}(y)\rangle^{m-k+\frac{n(1-p)}{n+p}} H_{n-1}^{\frac{p}{n+p}}(y)$$

$$\sum_{\substack{i_1,\dots,i_{n-1}\geq 0\\ i_1+2i_2+\cdots+(n-1)i_{n-1}=m}} c(n,p,i(m)) \prod_{j=1}^{n-1} \binom{n-1}{j}^{i_j} H_j^{i_j}(y)\, d\mathcal{H}^{n-1}(y)$$

$$= \lambda^{k-n\frac{n-p}{n+p}} \mathcal{W}^p_{m,\,k}(\lambda K).$$

Therefore,

$$\mathcal{V}^p_k(\lambda K) = \sum_{m=0}^{k} \binom{\frac{n(1-p)}{n+p}}{k-m} \mathcal{W}^p_{m,k}(\lambda K) = \sum_{m=0}^{k} \binom{\frac{n(1-p)}{n+p}}{k-m} \lambda^{k-n\frac{n-p}{n+p}} \mathcal{W}^p_{m,k}(K) = \lambda^{k-n\frac{n-p}{n+p}} \mathcal{V}^p_k(K).$$

Invariance

Theorem

Let K is a convex body in $\mathbb{R}^n$. Then for all $p \in \mathbb{R}$, $p \neq -n$, and for all $k \in \mathbb{N}$, $\mathcal{V}_k^p(K)$ (and $\mathcal{W}_{m,\,k}^p(K)$) are invariant under rotations and reflections.
When $p = 1$, $\mathcal{V}_k^1(K)$ (and $\mathcal{W}_{k,\,k}(K)$) are also invariant under translations.

Proof.

If T is a rotation or a reflection, then

- $|\det T| = 1$;
- $\|T^{-1t}(N_K(T^{-1}(y)))\| = \|N_K(T^{-1}(y))\| = 1$;
- $\{H_j(y) : y \in \partial T(K)\} = \{H_j(x) : x \in \partial K\}$ for all $1 \leq j \leq n-1$.

Applying the previous proposition, we get

$$\mathcal{W}_{m,\,k}^p(K) = \mathcal{W}_{m,\,k}^p(T(K)).$$

Thus,

$$\mathcal{V}_k^p(K) = \sum_{m=0}^{k} \binom{\frac{n(1-p)}{n+p}}{k-m} \mathcal{W}_{m,\,k}^p(K) = \sum_{m=0}^{k} \binom{\frac{n(1-p)}{n+p}}{k-m} \mathcal{W}_{m,\,k}^p(T(K)) = \mathcal{V}_m^p(T(K)).$$

Theorem

Let K is a convex body in $\mathbb{R}^n$. Then for $p \geq 0$ and $p < -n$, and for all $k \in \mathbb{N}$, $m \in \mathbb{N}$, $\mathcal{W}^p_{m,\,k}$ are valuations. Moreover, $\mathcal{V}^p_k$ are also valuations.

Remark. (when K is C^2_+)

Let K and L be convex bodies in $\mathbb{R}^n$ such that $K \cup L$ is a convex body. We need to show that

$$\mathcal{V}^p_k(K \cup L) + \mathcal{V}^p_k(K \cap L) = \mathcal{V}^p_k(K) + \mathcal{V}^p_k(L). \tag{1}$$

Using the fact that L_p affine surface area is a valuation,

$$as_p((K + tB^n_2) \cup (L + tB^n_2)) + as_p((K + tB^n_2) \cap (L + tB^n_2)) = as_p(K + tB^n_2) + as_p(L + tB^n_2),$$

$$(K + tB^n_2) \cup (L + tB^n_2) = (K \cup L) + tB^n_2 \quad \text{and} \quad (K + tB^n_2) \cap (L + tB^n_2) = (K \cap L) + tB^n_2$$

we get

$$as_p(K \cup L + tB^n_2) + as_p(K \cap L + tB^n_2) = as_p(K + tB^n_2) + as_p(L + tB^n_2).$$

After applying L_p Steiner formula for each term and collecting coefficients, we get (1).

Valuation: $p = 1$

Theorem

Let K is a convex body in $\mathbb{R}^n$. Then for all $k \in \mathbb{N}$, $m \in \mathbb{N}$, $\mathcal{V}_k^1 = \mathcal{W}_{k,\,k}$ are valuations.

We want to show that for convex bodies K and L in $\mathbb{R}^n$ such that $K \cup L$ is a convex body,

$$\mathcal{W}_{k,\,k}(K \cup L) + \mathcal{W}_{k,\,k}(K \cap L) = \mathcal{W}_{k,\,k}(K) + \mathcal{W}_{k,\,k}(L).$$

Note that

$$\mathcal{W}_{k,\,k}(K) = \int_{\partial K} H_{n-1}^{\frac{1}{n+1}}(x) \sum_{\substack{i_1,\dots,i_{n-1}\geq 0 \\ i_1+2i_2+\cdots+(n-1)i_{n-1}=k}} c(n,1,i(k)) \prod_{j=1}^{n-1} \binom{n-1}{j}^{i_j} H_j^{i_j}(x)\, d\mathcal{H}^{n-1}(x)$$

is a sum of (up to some constants) integrals of the form

$$\int_{\partial K} H_{n-1}^{\frac{1}{n+1}}(x) \prod_{i=1}^{j} k_{i_j}^{\alpha_j}(x)\, d\mathcal{H}^{n-1}(x).$$

Theorem

Let K be a convex body in $\mathbb{R}^n$. For all $1 \leq i_1, \ldots, i_j \leq n-1$ and $\alpha_1, \alpha_2, \ldots, \alpha_j \geq 0$,

$$\int_{\partial K} H_{n-1}^{\frac{1}{n+1}}(x) \prod_{i=1}^{j} k_{i_j}^{\alpha_j}(x)\, d\mathcal{H}^{n-1}(x)$$

is a valuation.

Let K and L be convex bodies in $\mathbb{R}^n$ such that $K \cup L$ is a convex body. We decompose a body using a strategy that was introduced by Schütt:

$$\begin{aligned}
\partial(K \cup L) &= \{\partial K \cap \partial L\} \cup \{\partial K \cap L^c\} \cup \{K^c \cap \partial L\} \\
\partial(K \cap L) &= \{\partial K \cap \partial L\} \cup \{\partial K \cap \text{int}(L)\} \cup \{\text{int}(K) \cap \partial L\} \\
\partial K &= \{\partial K \cap \partial L\} \cup \{\partial K \cap L^c\} \cup \{\partial K \cap \text{int}(L)\} \\
\partial L &= \{\partial K \cap \partial L\} \cup \{\partial L \cap K^c\} \cup \{\partial L \cap \text{int}(K)\}.
\end{aligned}$$

Valuation: $p = 1$

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\partial(K \cap L) &= \{\partial K \cap \partial L\} \cup \{\partial K \cap \operatorname{int}(L)\} \cup \{\operatorname{int}(K) \cap \partial L\} \\
\partial K &= \{\partial K \cap \partial L\} \cup \{\partial K \cap L^c\} \cup \{\partial K \cap \operatorname{int}(L)\} \\
\partial L &= \{\partial K \cap \partial L\} \cup \{\partial L \cap K^c\} \cup \{\partial L \cap \operatorname{int}(K)\}.
\end{aligned}$$

We also note that for all $x \in \partial K \cap \partial L$, and for all $\alpha \geq 0$ where the principal curvatures $k_j(\partial(K \cup L), x)$, $k_j(\partial(K \cap L), x)$, $k_j(K, x)$ and $k_j(L, x)$ exist for all $1 \leq i_1, \dots, i_j \leq n-1$,

$$\begin{aligned}
& H_{n-1}(\partial(K \cup L), x)^{\frac{1}{n+1}} k_j(\partial(K \cup L), x)^\alpha = \\
= \quad & \min\{H_{n-1}(K, x)^{\frac{1}{n+1}} k_j(K, x)^\alpha, H_{n-1}(L, x)^{\frac{1}{n+1}} k_j(L, x)^\alpha\}
\end{aligned}$$

and

$$\begin{aligned}
& H_{n-1}(\partial(K \cap L), x)^{\frac{1}{n+1}} k_j(\partial(K \cap L), x)^\alpha = \\
= \quad & \max\{H_{n-1}(K, x)^{\frac{1}{n+1}} k_j(K, x)^\alpha, H_{n-1}(L, x)^{\frac{1}{n+1}} k_j(L, x)^\alpha\}.
\end{aligned}$$

Continuity: general case

- $p \geq 1$: $as_p(K)$ is upper semi continuous *(Ludwig, Lutwak)*;
- $0 \leq p < 1$: $as_p(K)$ is upper semi continuous *(Ludwig, Hug)*;
- $-n < p < 0$: $as_p(K)$ is lower semi continuous *(Ludwig)*.

Question: are L_p-Steiner quermassintegrals $\mathcal{V}_k^p$ and L_p Steiner coefficients $\mathcal{W}_{m,k}^p$ (upper or lower semi) continuous?

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Question: are L_p-Steiner quermassintegrals $\mathcal{V}_k^p$ and L_p Steiner coefficients $\mathcal{W}_{m,k}^p$ (upper or lower semi) continuous?

Proposition

Let $p \neq -n$ and let $k \geq 1$. Then $\mathcal{V}_k^p$ are in general neither lower semi continuous nor upper semi continuous.

Proposition

Let $k \geq 1$. Then $\mathcal{V}_k^1 = \mathcal{W}_{k,\,k}$ are neither lower semi continuous nor upper semi continuous.

Continuity: $p = 1$

Proposition

Let $k \geq 1$. Then $\mathcal{V}_k^1 = \mathcal{W}_{k,\,k}$ are neither lower semi continuous nor upper semi continuous.

Consider convex bodies K_l, $l \in \mathbb{N}$:

$$K_l = \left(1 - \frac{1}{l}\right) B_\infty^n + \frac{1}{l} B_2^n,$$

where $B_\infty^n = \{x \in \mathbb{R}^n : \ ||x||_\infty \leq 1\}$ and B_2^n is the Euclidean ball. Then $K_l \to B_\infty^n$ in Hausdorff metric.

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Since $\mathcal{W}_{k,k}(B_\infty^n) = 0$,

$$\mathcal{W}_{k,\,k}(K_l) = \int\limits_{\partial K_l} H_{n-1}^{\frac{1}{n+1}} \sum_{\substack{i_1,\dots,i_{n-1} \geq 0 \\ i_1 + 2i_2 + \dots + (n-1)i_{n-1} = k}} c(n, 1, i(k)) \prod_{j=1}^{n-1} \binom{n-1}{j}^{i_j} H_j^{i_j}(x) \, d\mathcal{H}^{n-1}(x)$$

Proposition

Let $k \geq 1$. Then $\mathcal{V}_k^1 = \mathcal{W}_{k,\,k}$ are neither lower semi continuous nor upper semi continuous.

Consider convex bodies K_l, $l \in \mathbb{N}$:

$$K_l = \left(1 - \frac{1}{l}\right) B_\infty^n + \frac{1}{l} B_2^n,$$

where $B_\infty^n = \{x \in \mathbb{R}^n : \ ||x||_\infty \leq 1\}$ and B_2^n is the Euclidean ball. Then $K_l \to B_\infty^n$ in Hausdorff metric.
We denote by $B_2^n(x_0, r)$ the Euclidean ball with center at x_0 and radius r.

Since $\mathcal{W}_{k,k}(B_\infty^n) = 0$,

$$\mathcal{W}_{k,\,k}(K_l) = \int\limits_{\partial B_2^n\left(0,\frac{1}{l}\right)} H_{n-1}^{\frac{1}{n+1}} \sum_{\substack{i_1,\dots,i_{n-1}\geq 0 \\ i_1+2i_2+\cdots+(n-1)i_{n-1}=k}} c(n,1,i(k)) \prod_{j=1}^{n-1} \binom{n-1}{j}^{i_j} H_j^{i_j}(x)\, d\mathcal{H}^{n-1}(x)$$

$$= \mathcal{W}_{k,\,k}\left(B_2^n\left(0, \frac{1}{l}\right)\right) = l^{k-n\frac{n-1}{n+1}} \mathrm{Vol}_{n-1}\left(\partial B_2^n\right) C(n,1,k).$$

Therefore,

$$\mathcal{W}_{k,\,k}(K_l) = l^{k-n\frac{n-1}{n+1}} \mathrm{Vol}_{n-1}\left(\partial B_2^n\right) C(n,1,k),$$

where

$$C(n,1,k) = \binom{n\frac{n-1}{n+1}}{k}.$$

- If $k \geq n$ and
  - if $k - n + 1$ is even, then $C(n,1,k) > 0$ which implies that

    $$\mathcal{W}_{k,\,k}(K_l) = l^{k-n\frac{n-1}{n+1}} \mathrm{Vol}_{n-1}\left(\partial B_2^n\right) C(n,1,k) \to \infty \qquad \text{as } l \to \infty.$$

    Thus, $\mathcal{W}_{k,\,k}$ is not upper semi continuous (since $\mathcal{W}_{k,k}(B_\infty^n) = 0$).

    If we take a sequence of polytopes P_l that converge to B_2^n in Hausdorff metric, then $\mathcal{W}_{k,\,k}(P_l) = 0$, but $\mathcal{W}_{k,\,k}(B_2^n) = \mathrm{Vol}_{n-1}\left(\partial B_2^n\right)C(n,1,k) > 0$, so $\mathcal{W}_{k,\,k}$ is not lower semi continuous.

  - if $k - n + 1$ is odd, then then $C(n,1,k) < 0$ which similarly implies that $\mathcal{W}_{k,\,k}$ is not upper or lower semi continuous.

- if $k \leq n-1$, $\mathcal{W}_{k,\,k}$ is not lower semi continuous since $C(n,1,k) > 0$ and not upper semi continuous.

Continuity: mixed affine surface areas

For all $p \neq -n$ and all $s \in \mathbb{R}$, the s-th mixed L_p affine surface area of K (Lutwak (1987) for $p \geq 1$ and all $s \in \mathbb{R}$; Werner, Ye (2010) for all $p \neq -n$ and all s) is defined as

$$as_{p,\,s}(K) = \int\limits_{\partial K} H_{n-1}(x)^{\frac{s+p}{n+p}} \langle x, N(x)\rangle^{(1-p)\,\frac{n-s}{n+p}} \, d\mathcal{H}^{n-1}(x).$$

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Special cases of the L_p Steiner coefficients:

- $k = m = l(n-1)$, $l \in \mathbb{N}$ and $p = 1$

$$\mathcal{V}^1_{l(n-1)}(K) = \mathcal{W}_{l(n-1),\, l(n-1)}(K) = \binom{\frac{n}{n+1}}{l} as_{1, l(n+1)}(K).$$

- $m = 0$

$$\mathcal{W}^p_{0,\,k}(K) = \int_{\partial K} \langle x, N(x)\rangle^{-k+\frac{n(1-p)}{n+p}} H_{n-1}^{\frac{p}{n+p}} \, d\mathcal{H}^{n-1}(x) = as_{p+\frac{k}{n}(n+p),\,-k}(K).$$

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- $m = 0$

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Theorem

For $p \geq 0$, $\mathcal{W}^p_{0,\,k} = as_{p+\frac{k}{n}(n+p),-k}$ are upper semi continuous $n\frac{n-p}{n+p} - k$ homogeneous valuations that are invariant under rotation and reflections.

Thank you!