Fonds National de la Recherche Luxembourg

UNIVERSITÉ DU LUXEMBOURG

Variations on a Theorem by Peter de Jong

Giovanni Peccati (Luxembourg University)

Paris, IHP — June 7th, 2022

INTRODUCTION

- ⋆ **Topic**: multivariate and functional fluctuations of ***U*-statistics**.

- ⋆ *In the last 15 years*: proof of several **fourth moment theorems** for sequences of random variables living in eigenspaces of Markov operators, e.g.: **Gaussian Wiener chaos** (Nualart & Peccati, 2005; Nourdin & Peccati, 2007–10); **Poisson Wiener chaos** (Peccati, Solé, Taqqu & Utzet, 2010; Döbler & Peccati, 2018); **diffusive Markov operators** (Ledoux, 2010; Azmoodeh, Campese & Poly, 2013).

- ⋆ **Applications**: mathematical statistics, mathematical physics, stochastic geometry (random geometric graphs & geometry of random fields), ...

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Framework, I

- ⋆ Let $(X_1, \ldots, X_n)$ be independent random elements with values in $(E, \mathcal{E})$.
- ⋆ For $k = 1, \ldots, n$, a (square-integrable) **U-statistic** of order k is a random variable with the form

$$W = \sum_{1 \le i_1 < \cdots < i_k \le n} g_{(i_1,\ldots,i_k)}(X_{i_1}, \ldots, X_{i_k}),$$

  with $g_{(i_1,\ldots,i_k)} : E^k \to \mathbb{R}$ square-integrable.
- ⋆ W is **symmetric** if the X_i's are **i.i.d.** and $g_{(i_1,\ldots,i_k)} \equiv g$ is symmetric.
- ⋆ W is **degenerate** if

$$\mathbb{E}[g_{(i_1,\ldots,i_k)}(X_{i_1}, \ldots, X_{i_k}) \mid X_a : a \in A] = 0,$$

  for all $A \subsetneq \{i_1, \ldots, i_k\}$.

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- ⋆ **Fact**: every square-integrable $F = F(X_1, ..., X_n)$ can be uniquely decomposed into a sum of degenerate U-statistics of order $k = 0, 1, ..., n$ (**Hoeffding decomposition**).
- ⋆ In the case where $X_1, ..., X_n$ are centered and real-valued, classical examples of *non-symmetric and degenerate* U-statistics are **homogeneous sums**:

$$W = \sum_{1 \le i_1 < \cdots < i_k \le n} a_{(i_1,...,i_k)} X_{i_1} \cdots X_{i_k}, \quad a_{(i_1,...,i_k)} \in \mathbb{R}.$$

- ⋆ For **Rademacher variables**: "degenerate U-statistics of order k" ⇔ "homogeneous sums of order k" ⇔ "**kth Walsh chaos**".
- ⋆ The **maximal influence** associated with a degenerate W is

$$\mathrm{Inf}(W) = \max_{i=1,...,n} \sum_{i \in \{i_1,...,i_k\}} \mathbb{E}[g_{(i_1,...,i_k)}(X_{i_1}, ..., X_{i_k})^2]$$

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Main Theme: Peter de Jong, 1990

Journal of Multivariate Analysis 34, 275–289 (1990)

A Central Limit Theorem for Generalized Multilinear Forms

Peter de Jong

Courseware Europe b.v., Ebbehout 1, 1507 EA Zaandam, The Netherlands

Communicated by the Editors

Let $X_1, ..., X_n$ be independent random variables and define for each finite subset $I \subset \{1, ..., n\}$ the σ-algebra $\mathscr{F}_I = \sigma\{X_i : i \in I\}$. In this paper $\mathscr{F}_I$-measurable random variables W_I are considered, subject to the centering condition $E(W_I \mid \mathscr{F}_J) = 0$ a.s. unless $I \subset J$. A central limit theorem is proven for d-homogeneous sums $W(n) = \sum_{|I|=d} W_I$, with var $W(n) = 1$, where the summation extends over all $\binom{n}{d}$ subsets $I \subset \{1, ..., n\}$ of size $|I| = d$, under the condition that the normed fourth moment of $W(n)$ tends to 3. Under some extra conditions the condition is also necessary.

1. Introduction and Summary

We start with a sketch of the general setting. Consider independent random variables $X_1, ..., X_n$ on the probability space $(\Omega, \mathscr{F}, P)$. Define for each finite subset $I \subset \{1, ..., n\}$ the σ-algebra $\mathscr{F}_I = \sigma\{X_i : i \in I\}$ (with $\mathscr{F}_\varnothing$ the trivial σ-algebra) and let W_I denote an $\mathscr{F}_I$-measurable random variable. (Throughout this paper the random variables W_I may depend on n, $W_I = W_{In}$; the parameter n will be suppressed where possible.) We assume the random variables W_I to be centered, square integrable, and uncorrelated:

$$EW_I = 0, \qquad EW_I^2 = \sigma_I^2 < \infty, \qquad EW_I W_J = 0 \quad \text{if} \quad I \neq J.$$

In a beautiful 1990 paper, **Peter de Jong** proved a surprising result for a sequence $\{W_n : n \geq 1\}$ of normalised **degenerate U-statistics**, that is:

If $\mathrm{Inf}(W_n) \to 0$, *then* W_n **verifies a Central Limit Theorem** *provided*

$$\mathbb{E}W_n^4 \to 3 \; (= \mathbb{E}N(0,1)^4).$$

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Remarks

- ⋆ *Original proof*: **martingale CLT** + (heavy) combinatorial analysis.

- ⋆ In *Döbler & Peccati* (EJP, 2016) (using **Stein's method**): *Let* $\mathbf{W}_n = (W_n^{(1)}, ..., W_n^{(d)})$*,* $n \geq 1$*, be vectors of degenerate U-statistics such that* (i) $\mathrm{Cov}(\mathbf{W}_n) \to \Sigma$, (ii) $\max_i \mathrm{Inf}(W_n^{(i)}) \to 0$, (iii) $\mathbb{E}[(W_n^{(i)})^2 (W_n^{(j)})^2] \to \Sigma(i,i)\Sigma(j,j) + 2\Sigma(i,j)^2$.
  Then, $\mathbf{W}_n \Rightarrow \mathrm{N}_d(0, \Sigma)$ + **quantitative bounds**.

- ⋆ *Combinatorial component of 2016 proofs*: basically, **unchanged since de Jong (1990)**.

- ⋆ Careful bookkeeping of constants allows for $d = d_n \to \infty$ as soon as $d_n! \ll \max_\ell \mathrm{Inf}(W_n^{(\ell)})^\alpha$ (some $\alpha > 0$; *absolutely not sharp*).

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SYMMETRIC U-PROCESSES

- ⋆ Let $\{X_i : i \geq 1\}$ be a sequence of **i.i.d. random elements**.
- ⋆ Fix $k \geq 2$, and let $g_n : E^k \to \mathbb{R}$ be a sequence of **symmetric, square-integrable and degenerate kernels**.
- ⋆ The **sequential U-process** associated with g_n is the random function

$$t \mapsto U(g_n; t) := \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq \lfloor nt \rfloor} g_n(X_{i_1}, ..., X_{i_k}), \ t \in [0,1].$$

- ⋆ **Problem**: study the convergence of $U(g_n; \cdot)$ in the **Skorohod space** $D[0,1]$. We write $U_n := U(g_n; 1)$, and $\sigma_n^2 := \mathrm{Var}(U_n)$.

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Statement (Executive Summary)

Döbler, Kasprzak & Peccati (AAP, 2022)

Let $\widetilde{U}(g_n;\cdot) := U_n(g_n;\cdot)/\sigma_n$ *and* $\widetilde{U}_n := U_n/\sigma_n$. *Assume that*

(a) $\mathrm{Inf}(\widetilde{U}_n) \to 0$;

(b) $\left|\mathbb{E}[\widetilde{U}_n^4] - 3\right| \ll n^{-\eta}$ *for some* $\eta > 0$ + *"technicalities".*

Then,

$$\{U(g_n;t) : t \in [0,1]\} \Longrightarrow \{B(t^k) : t \in [0,1]\},$$

in $D[0,1]$, *where* B *is a standard Brownian motion.*

Remarks: (i) Condition (b) is checked by using "contraction operators"
(ii) A perfectly working multivariate version is also available.

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Generalizing Miller and Sen (1972)

⋆ Consider a normalized, non-degenerate (centered and symmetric) U-process U_n on $[0,1]$, along with its Hoeffding decomposition:

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$$U_n \Longrightarrow \left\{ \sum_{k=1}^{p} b_k^2 \times t^{p-k} B_k(t^k), \quad t \in [0,1] \right\},$$

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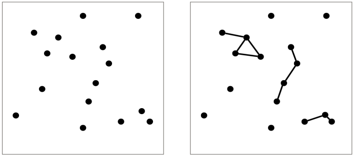
Example (Changepoint Analysis)

- ⋆ Let $r_n \downarrow 0$ Consider $X_1, ..., X_n$ i.i.d. uniform points on the unit cube $\subset \mathbb{R}^d$, and connect X_ℓ and X_j if $0 < \|X_\ell - X_j\| < r_n$:



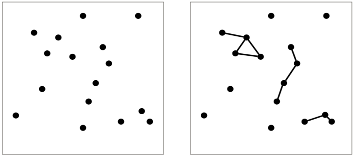
- ⋆ We are interested in the **changepoint empirical process**

$$S_n(t) := \sum_{1 \leq i \leq \lfloor nt \rfloor < j \leq n} \mathbf{1}_{\{X_i \sim X_j\}}, \quad t \in [0,1].$$

- ⋆ Instance of **graph-based changepoint detection**: see Chen & Zhang (2015)

Example (Changepoint Analysis)

- ⋆ Let $r_n \downarrow 0$ Consider $X_1, ..., X_n$ i.i.d. uniform points on the unit cube $\subset \mathbb{R}^d$, and connect X_ℓ and X_j if $0 < \|X_\ell - X_j\| < r_n$:



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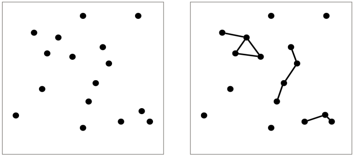

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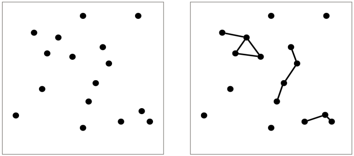

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Example (Changepoint Analysis)

Döbler, Kasprzak & Peccati (AAP, 2022)

Consider the **sparse regime**: $nr_n^d \to 0$ *and* $n^2 r_n^d \to \infty$. *Then, there exists a constant* $c > 0$ *(depending on* d*) such that, setting* $\sigma_n^2 := c\, n^2 r_n^d$, *and*

$$T_n(t) := \frac{S(n,t) - \mathbb{E}[S(n,t)]}{\sigma_n}, \quad t \in [0,1],$$

one has that, if $n^{2-\delta} \ll r_n^d \ll n^{-1}$ *for some* $\delta > 0$, *then*

$$T_n \Longrightarrow \{\sqrt{2}\, b(t) : t \in [0,1]\},$$

where b *is a standard Brownian bridge. In particular:*

$$\arg\max_{t \in [0,1]} (-T_n(t)) \Longrightarrow \mathbf{U}_{[0,1]}.$$

HOMOGENOEUS SUMS: DE JONG & UNIVERSALITY

Nourdin, Peccati & Reinert (AoP, 2010)

Let $\mathbf{G} = \{G_i : i \geq 1\}$ *be i.i.d.* $N(0,1)$*. Consider a sequence of normalized homogeneous sums or order* $k \geq 2$*:*

$$Q_n(\mathbf{G}) := \sum_{1 \leq i_1 < i_2 < \cdots i_k \leq n} a^{(n)}_{(i_1,\dots,i_k)} G_{i_1} \cdots G_{i_k}, \quad n \geq 1.$$

If $\mathbb{E}Q_n(G)^4 \to 3$*, then*

$$Q_n(\mathbf{X}) = \sum_{1 \leq i_1 < i_2 < \cdots i_k \leq n} a^{(n)}_{(i_1,\dots,i_k)} X_{i_1} \cdots X_{i_k} \Rightarrow N(0,1),$$

for every sequence $\mathbf{X} := \{X_i : i \geq 1\}$ *of independent centered r.v.'s such that* $\sup_i \mathbb{E}|X_i|^{2+\epsilon} < \infty$.

Recent applications: Caravenna, Sun & Zygouras (2016++, polymers); Angst & Poly (2021, zeros of random polynomials).

Homogenoeus Sums: de Jong & Universality

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Döbler, Kasprzak & Peccati (PTRF, 2022+)

Consider **sequential U-processes** *associated with normalized non-symmetric U-statistics:*

$$t \mapsto U_n(t) := \sum_{1 \le i_1 < i_2 < \cdots < i_k \le \lfloor nt \rfloor} g^{(n)}_{(i_1,\dots,i_k)}(X_{i_1}, \dots, X_{i_k}), \ t \in [0,1].$$

Assume that

(a) $\mathrm{Inf}(U_n(1)) \to 0$;

(b) $\mathbb{E}[U_n(1)^4] \to 3$;

(c) $\mathbb{E}[g^{(n)}_{(i_1,\dots,i_k)}(X_{i_1}, \dots, X_{i_k})^4] \le C\mathbb{E}[g^{(n)}_{(i_1,\dots,i_k)}(X_{i_1}, \dots, X_{i_k})^2]^2$

Then, $\{U_n\}$ is relatively compact in $D[0,1]$ and every adherent point corresponds to the law of a **continuous Gaussian process**.

Döbler, Kasprzak & Peccati (PTRF, 2022+)

Let $\mathbf{G} = \{G_i : i \geq 1\}$ *be i.i.d.* $N(0,1)$. *Consider a sequence of normalized homogeneous U-processes:*

$$Q_{\lfloor nt \rfloor}(\mathbf{G}) := \sum_{1 \leq i_1 < i_2 < \cdots i_k \leq \lfloor nt \rfloor} a^{(n)}_{(i_1,\ldots,i_k)} G_{i_1} \cdots G_{i_k}, \quad t \in [0,1].$$

If $Q_{\lfloor nt \rfloor}(\mathbf{G})$ *converges in* $D[0,1]$ *to a continuous Gaussian process, then the same holds for*

$$Q_{\lfloor nt \rfloor}(\mathbf{X}) = \sum_{1 \leq i_1 < i_2 < \cdots i_k \leq \lfloor nt \rfloor} a^{(n)}_{(i_1,\ldots,i_k)} X_{i_1} \cdots X_{i_k},$$

for every sequence $\mathbf{X} := \{X_i : i \geq 1\}$ *of independent centered r.v.'s such that* $\sup_i \mathbb{E}|X_i|^4 < \infty$.

Fractional Products

- ⋆ Fix $k \geq 3$, as well as $m = 2, ..., k-1$.
- ⋆ Write $N = n^m$, and let $\varphi : [n]^m \to [N]$ be one-to-one.
- ⋆ Consider a **connected m-cover** $S_1, ..., S_k$ of $[k]$, that is: (i) $\cup_i S_i = [k]$, (ii) $|S_i| = m$ and (iii) each index $i \in [k]$ appears in exactly m subsets S_i.
- ⋆ We define

$$F_N := \{(\varphi(\pi_{S_1}\mathbf{a}), ..., \varphi(\pi_{S_k}\mathbf{a})) : \mathbf{a} \in [n]^k\},$$

and denote by $\tilde{F}_N$ its symmetrization.

- ⋆ Then $\tilde{F}_N$ is a symmetric subset of $[N]^k$ s.t. $|\tilde{F}_N| \asymp N^{k/m}$ (**Fractional Cartesian Product**)

Fractional Products

Döbler, Kasprzak & Peccati (PTRF, 2022+)

For every sequence $\mathbf{X} := \{X_i : i \geq 1\}$ *of independent centered r.v.'s such that* $\sup_i \mathbb{E}|X_i|^4 < \infty$, *the empirical process*

$$Q_{\lfloor Nt \rfloor}(\mathbf{X}) = \frac{1}{|\tilde{F}_N|^{1/2}} \sum_{1 \leq i_1 < i_2 < \cdots i_k \leq \lfloor Nt \rfloor} \mathbf{1}_{\{i_1,\ldots,i_k\} \in \tilde{F}_N} X_{i_1} \cdots X_{i_k},$$

weakly converges to a multiple of $\{B(t^{k/m}) : t \in [0,1]\}$, *where B is a standard Brownian motion* (*).

(*) *Not* achievable by symmetric statistics.

Final Words

THANK YOU FOR YOUR ATTENTION!