Criteria for entropic curvature on graphs along Schrödinger bridges at zero temperature.

- Entropic curvature along Schrödinger bridges at zero temperature, ArXiv
- Forthcoming preprint : joint work with Martin Rapaport

Institut Henri Poincaré, Paris

Phenomena in High Dimension

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The entropic curvature of $(\mathcal{X}, d, m)$ is lower bounded by $K \in \mathbb{R}$ if for any $\nu_0, \nu_1 \in \mathcal{P}_2(\mathcal{X})$, there exists a constant speed W_2-geodesic $(\nu_t)_{t\in[0,1]}$ from ν_0 to ν_1 such that for all $t \in [0,1]$,

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(1) $\Leftrightarrow$ Bakry-Emery curvature condition $CD(K, \infty)$: $\mathrm{Ricc} + \mathrm{Hess(V)} \geqslant \mathrm{K}$.

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- many other papers dealing with Erbar-Maas entropic approach,
and also many recent papers dealing with Bakry-Emery conditions on graphs.

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$(\widehat{Q}_t^\gamma)_{t \in [0,1]}$: a Schrödinger bridge from ν_0 to ν_1, an entropic interpolation.

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It implies as $\gamma \to 0$,

$$\widehat{Q}_t(z) = \sum_{x,y\in\mathcal{X}} Q_t^{x,y}(z)\, \widehat{\pi}(x,y), \qquad \text{with} \qquad \iint d(x,y)\, d\widehat{\pi}(x,y) = W_1(\nu_0,\nu_1).$$

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$$Q_t^{x,y}(z) := \mathbb{1}_{[x,y]}(z)\, r(x,z,y)\, \rho_t^{d(x,y)}(d(x,z)), \quad z\in\mathcal{X},$$

with for any $x,z,y\in\mathcal{X}$,

- $[x,y]$ is the set of all points that belongs to a discrete geodesic from x to y,
- $r(x,z,y) := \dfrac{L^{d(x,z)}(x,z)L^{d(z,y)}(z,y)}{L^{d(x,y)}(x,y)}$,
- ρ_t^d denotes the binomial law with parameters $t\in[0,1]$ and $d\in\mathbb{N}$:

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Structure of the "Schrödinger briges at zero temperature"

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where $(Q_t^{x,y})_{t\in[0,1]}$ is the binomial path from δ_x to δ_y defined by

$$Q_t^{x,y}(z) := \mathbb{1}_{[x,y]}(z)\, r(x,z,y)\, \rho_t^{d(x,y)}(d(x,z)), \quad z \in \mathcal{X},$$

with for any $x, z, y \in \mathcal{X}$,

- $[x, y]$ is the set of all points that belongs to a discrete geodesic from x to y,
- $r(x,z,y) := \dfrac{L^{d(x,z)}(x,z)L^{d(z,y)}(z,y)}{L^{d(x,y)}(x,y)}$,
- ρ_t^d denotes the binomial law with parameters $t \in [0,1]$ and $d \in \mathbb{N}$:

$$\rho_t^d(k) := \binom{d}{k} t^k (1-t)^{d-k}, \quad k \in \{0,\dots,d\}.$$

Interpretation : Let $d = d(x,y)$ and N_t be a binomial random variable with law ρ_t^d. Let $\Gamma = (\Gamma_0,\dots,\Gamma_d)$ be a random discrete geodesic from x to y, independent of N_t with law

$$\mathbb{P}(\Gamma = \alpha) = \frac{L(\alpha_0,\alpha_1)\cdots L(\alpha_{d-1},\alpha_d)}{L^{d(x,y)}(x,y)}, \qquad \text{if } \alpha = (\alpha_0,\alpha_1,\dots,\alpha_d).$$

Then $\widehat{Q}_t$ is the law of Γ_{N_t}.

Definition : C-displacement convexity property of entropy

On the discrete space $(\mathcal{X}, d, m, L)$, one says that the relative entropy is C-displacement convex where $C = (C_t)_{t\in[0,1]}$, if for any probability measure $\nu_0, \nu_1 \in \mathcal{P}_b(X)$, the Schrödinger bridge at zero temperature $(\widehat{Q}_t)_{t\in[0,1]}$ from ν_0 to ν_1, satisfies for any $t \in (0,1)$,

$$H(\widehat{Q}_t|m) \leqslant (1-t)H(\nu_0|m) + t\,H(\nu_1|m) - \frac{t(1-t)}{2} C_t(\widehat{\pi}). \qquad (2)$$

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- Similarly, one defines the *T_2-entropic curvature* of the space $(\mathcal{X}, d, m, L)$ as the largest constant $K_2 \in \mathbb{R}$ so that (2) holds for any ν_0, ν_1, t with

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- One may consider other costs, for example $C_t(\widehat{\pi}) \geqslant \widetilde{K}\, \widetilde{T}_2(\widehat{\pi})$

$$\widetilde{T}_2(\widehat{\pi}) := \left[\int \left(\int d(x,y)\, d\widehat{\pi}_{\rightarrow}(y|x) \right)^2 d\nu_0(x) + \int \left(\int d(x,y)\, d\widehat{\pi}_{\leftarrow}(x|y) \right)^2 d\nu_1(y) \right],$$

where $\widehat{\pi}(x,y) = \nu_0(x)\widehat{\pi}_{\rightarrow}(y|x) = \nu_1(y)\widehat{\pi}_{\leftarrow}(x|y)$.

Geometric conditions on balls of radius 2 for entropic curvature

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- If $W \subset W' \subset S_2(z)$ then $\kappa(z, W) \leqslant \kappa(z, W') \leqslant \kappa(z, S_2(z))$.
- If for some $z_0'' \in S_2(z)$, $S_1(z) \cap [z, z_0''] = \{z_0'\}$, then

$$\kappa(z, S_2(z)) \geqslant \kappa(z, \{z_0''\}) = \sup_{\alpha} \alpha(z_0')^2 = 1.$$

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A Bonnet-Myers Theorem

If the space $(\mathcal{X}, d, m, L)$ has positive T_2-entropic curvature, then its diameter is bounded. Therefore, the assumptions imply that $\mathcal{X}$ is finite.

Assume that $\kappa_L < 1$. For $z \in \mathcal{X}$, let

$$c_L(z) := \left(\sup_{W_+, W_-} \left\{ \frac{\mathbb{1}_{W_+ \neq \emptyset}}{1 - \kappa_L(z, W_+)} + \frac{\mathbb{1}_{W_- \neq \emptyset}}{1 - \kappa_L(z, W_-)} \right\} \right)^{-1},$$

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Theorem

If $\kappa_L < 1$ then the space $(\mathcal{X}, d, m, L)$ has positive T_2-entropic curvature and also positive W_1-entropic curvature, more precisely

$$C_t(\widehat{\pi}) \geqslant 4c_L \max\left\{ W_1(\nu_0, \nu_1)^2, \iint c_2(d(x,y))\, d\widehat{\pi}(x,y) \right\},$$

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If $\kappa_L < 1$ then the space $(\mathcal{X}, d, m, L)$ has positive T_2-entropic curvature and also positive W_1-entropic curvature, more precisely

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with $c_2(d) := \max\left\{ \frac{d(d-1)}{2}, d^2 - 2d(1 + \log d)\mathbb{1}_{d \neq 0} \right\}, \quad d \in \mathbb{N}.$

Assume that $\kappa_L < 1$. For $z \in \mathcal{X}$, let

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where the supremum runs over all $W_+, W_- \subset S_2(z)$,

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Assume that on the space $(\mathcal{X}, d, m, L)$, the C-displacement convexity property of entropy holds with $C_t(\widehat{\pi}) \geqslant K_2 \iint c_2(d(x,y))\, d\widehat{\pi}(x,y)$.

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Proof : Choose $\nu_0 = \mu_f$ and let t go to 0 in the C-displacement convexity property.

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For any functions f, g, h, k on $\mathbb{Z}$ satisfying

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one has $$\left(\int_{\mathbb{Z}} e^f\, dm_0\right)\left(\int_{\mathbb{Z}} e^g\, dm_0\right) \leqslant \left(\int_{\mathbb{Z}} e^k\, dm_0\right)\left(\int_{\mathbb{Z}} e^h\, dm_0\right).$$

Also consequence of a "convex entropy inequality" :

Theorem : Gozlan-Roberto-S.-Tetali (20'). Halikias-Klartag-Slomka (21'), Slomka (21')

$$H(\nu_-|m_0) + H(\nu_+|m_0) \leqslant H(\nu_0|m_0) + H(\nu_1|m_0),$$
where $\nu_- = m_{-\#}\widehat{\pi}$, $\quad \nu_+ = m_{+\#}\widehat{\pi}$, $\quad \widehat{\pi}$ monotone coupling of ν_0 and ν_1.

Examples of graphs

- The lattice $\mathbb{Z}^n = \mathcal{X}$ with counting measure m_0

$(\widehat{Q}_t)_{t\in[0,1]}$: the Schrödinger bridge at zero temperature joining ν_0 to ν_1

$$\widehat{Q}_t(z) = \sum_{x,y\in\mathcal{X}} Q_t^{x,y}(z)\,\widehat{\pi}(x,y),$$

where for $x = (x_1,\ldots,x_n)$, $y = (y_1,\ldots,y_n)$,

$$Q_t^{x,y}(z) = \binom{|y_1 - x_1|}{|z_1 - x_1|}\cdots\binom{|y_n - x_n|}{|z_n - x_n|}\, t^{d(x,z)}(1-t)^{d(z,y)}\mathbb{1}_{[x,y]}(z).$$

For all $z \in \mathbb{Z}^n$, $\quad \kappa(z, S_2(z)) = 1 \Rightarrow K_2 \geqslant 0$.

Bonney-Myers Theorem implies $K_2 = 0$, a result by E. Hillion (14').

Another Prékopa-Leindler inequality on $\mathbb{Z}$ for $t = 1/2$,

Theorem : Klartag-Lehec (19')

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As a consequence, if $\mu_0 = m_0/m_0(\mathcal{X})$ then

$$\frac{2}{n}\inf_{\pi\in\Pi(\nu_0,\nu_1)} \iint c_2(d(x,y))\, d\pi(x,y) \leqslant \left(\sqrt{H(\nu_0|\mu_0)} + \sqrt{H(\nu_1|\mu_0)}\right)^2,$$

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with $c_2(d) \underset{+\infty}{\sim} d^2$. Using the CLT, implies Talagrand's inequality for the standard Gaussian measure, γ_o : for any $\nu_0, \nu_1 \in \mathcal{P}_2(\mathbb{R})$

$$\frac{1}{2} W_2^2(\nu_0,\nu_1) \leqslant \left(\sqrt{H(\nu_0|\gamma_o)} + \sqrt{H(\nu_1|\gamma_o)}\right)^2.$$

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from which we get $\varphi''(t) \geqslant 2\lambda_{\min}(A) \iint d(x,y)\, d\widehat{\pi}(x,y)$, with $\lambda_{\min}(A) \leqslant 0$.

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from which we get $\varphi''(t) \geqslant 2\lambda_{\min}(A) \iint d(x,y)\, d\widehat{\pi}(x,y)$, with $\lambda_{\min}(A) \leqslant 0$.
It follows that $H(\cdot|\mu_V)$ satisfies the C-displacement convexity property with

$$C_t(\widehat{\pi}) \geqslant \frac{2}{n} \iint d(x,y)(d(x,y)-1) d\widehat{\pi}(x,y) + 2\lambda_{\min}(A) \iint d(x,y)\, d\widehat{\pi}(x,y).$$

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m_0 is the counting measure on a slice of the discrete cube $\{0,1\}^n$ of order $k \in [n]$:

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A lower bound of order $Cste/n$ could be expected, du to known W_1-transport entropy inequality and modified Logarithmic Sobolev inequality for $\mu_0 = m_0/m_0(S_n)$.

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- One may also consider graphs with non-positive T_2-entropic curvature : The so-called geodetic graphs (only one discrete geodesic between two vertices) like the trees :

$$-2\log\left(1 + [\max_{z\in\mathcal{X}} \mathrm{Deg}(z) - 2]_+\right) \leqslant K_2 \leqslant 0.$$

Open questions - work in progress :

- Find new connections between curvature along W_1-geodesics and modified logarithmic Sobolev inequalities for Cayley graphs.
- Consider measures on graphs with potential interactions.
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