Variance conjecture in Schatten balls

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Phenomena in High Dimension
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Outline

Introduction and results

Proof summary of Theorems A and B

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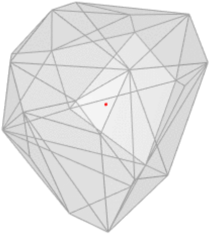

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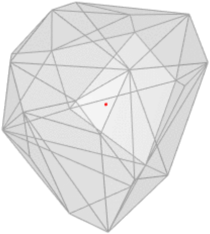

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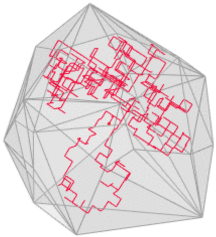

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Hence for n large, $\operatorname{Var}_K |\boldsymbol{x}|^2 \le \frac{1}{2} \cdot \frac{\big(\operatorname{Tr} \Sigma_K\big)^2}{d_n} \le \frac{1}{2} \|\Sigma_K\|_{\mathsf{op}} \operatorname{Tr} \Sigma_K$.

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- Case $p = \infty$ proved by Radke and Vritsiou (2020).

General statements

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For either $E = \mathcal{M}_n(\mathbb{F})$ or $E = \{T = T^*\}$, for any $p \in [1, \infty)$ and $q > 0$,

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Theorem B (Asymptotics of the q-inertia moments).
For $E = \{T = T^*\}$, $p \in (3, \infty)$ and any $q > 0$,

$$\frac{I_q(K)}{I_2(K)} = 1 + \frac{(q-2)(p-2)^2}{16p^2\, d_n} + o\left(\frac{1}{d_n}\right).$$

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Theorem 1 follows by taking $q = 4$ and raising to the fourth power.

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$$\overset{T=USV^*}{=} \frac{c_n}{\left|B_E\left(S_p^n\right)\right|} \int_{\|\boldsymbol{s}\|_p \leq 1} |\boldsymbol{s}|^q \, f_{a,b,c}(\boldsymbol{s}) \, \mathrm{d}\boldsymbol{s}$$

where c_n is explicit and depend on $|U_n(\mathbb{F})|$ and E, and

$$f_{a,b,c}(\boldsymbol{s}) := \prod_{1 \leq i < j \leq n} |s_i^a - s_j^a|^b \prod_{i=1}^{n} |s_i|^c,$$

with $b := \dim_{\mathbb{R}} \mathbb{F}$, $a \in \{1, 2\}$, $c \in \{0, 1, 3\}$ depending on E and $\mathbb{F}$.

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$$\mathbb{E}_K |\boldsymbol{x}|^q = \frac{1}{\left|B_E\left(S_p^n\right)\right|} \int_{B_E\left(S_p^n\right)} |\boldsymbol{s}(T)|^q \, \mathrm{d}T$$

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where c_n is explicit and depend on $|U_n(\mathbb{F})|$ and E, and

$$f_{a,b,c}(\boldsymbol{s}) \; := \prod_{1 \le i < j \le n} |s_i^a - s_j^a|^b \prod_{i=1}^n |s_i|^c,$$

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- Kabluchko-Prochno-Thäle '19,'20: $|B_E(S_p^n)|^{1/d_n}$ & WLLN for $\boldsymbol{s}(T)$.

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We introduce

$$\mathbb{P}_{n,p}(\mathrm{d}\boldsymbol{x}) := \frac{1}{Z_{n,p}}\, f_{a,b,c}(\boldsymbol{x})\ \mathrm{e}^{-abn\gamma_p\|\boldsymbol{x}\|_p^p}\,\mathrm{d}\boldsymbol{x}. \qquad \left(\gamma_p := \tfrac{1}{2}B\left(\tfrac{p}{2},\tfrac{1}{2}\right)\right)$$

Reduction to linear statistics of β-ensembles

$$\mathbb{E}_K |\boldsymbol{x}|^q = \frac{Z_{n,p}\, c_n (abn\gamma_p)^{\frac{d_n+q}{p}}}{\left|B_E\left(S_p^n\right)\right| \Gamma\left(1+\frac{d_n+q}{p}\right)} \int_{\mathbb{R}^n} |\boldsymbol{x}|^q \,\mathbb{P}_{n,p}(\mathrm{d}\boldsymbol{x}). \qquad (L_q)$$

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$$\frac{I_q(K)}{\sqrt{d_n}} = \frac{\Gamma\left(1+\frac{d_n}{p}\right)^{\frac{1}{q}+\frac{1}{d_n}}}{\Gamma\left(1+\frac{d_n+q}{p}\right)^{\frac{1}{q}} c_n{}^{\frac{1}{d_n}}} \cdot \frac{\left(\mathbb{E}_{n,p} |\boldsymbol{x}|^q\right)^{\frac{1}{q}}}{Z_{n,p}{}^{\frac{1}{d_n}}}.$$

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First order asymptotics

Let $V_p(x) := 2\gamma_p|x|^p$ and $L_n(\boldsymbol{x}) := n^{-1}\sum_{i=1}^n \delta_{x_i}$.

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$$= \frac{1}{Z_{n,p}} \, \mathrm{e}^{-\frac{bn^2}{2} H_{n,p}(\boldsymbol{x})} \, \mathrm{d}\boldsymbol{x},$$

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minimal when $L_n(\boldsymbol{x}) \in \operatorname*{argmin}_{\mathcal{P}_1(\mathbb{R})} \mathcal{I}_p$

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- $\underset{\mathcal{P}_1(\mathbb{R})}{\operatorname{argmin}} \mathcal{I}_p = \{\mu_p\}$, $\mu_p \ll \mathrm{Leb}$ and $\operatorname{Supp}\mu_p = [-1, 1]$;
- $\mathbb{E}_{n,p}\langle L_n(\boldsymbol{x}), f\rangle \xrightarrow[n\to\infty]{} \langle \mu_p, f\rangle$ for all $f \in \mathcal{C}_b$;
- $-\frac{1}{d_n}\log Z_{n,p} \xrightarrow[n\to\infty]{} \mathcal{I}_p(\mu_p) = \log 2 + \frac{3}{2p}$.

[Saff-Totik '97,
Ben Arous-Guionnet '97,
Johansson '98,
Hiai-Petz '00]

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Let $L_{n,p}$ be a random measure having same law as $\dfrac{1}{n}\sum_{i=1}^{n}\delta_{x_i}$ under $\mathbb{P}_{n,p}$.

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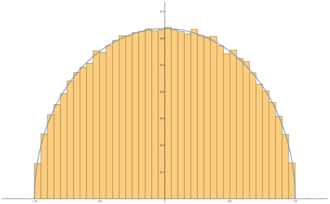
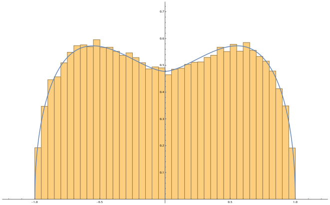
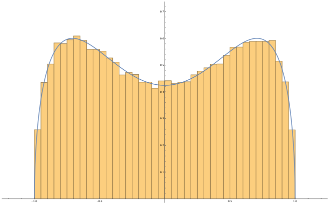


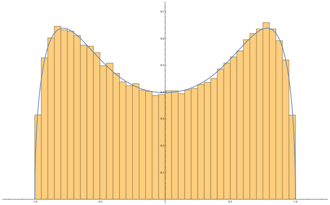
(A) $p=2$;



(B) $p=3$;



(C) $p=4$;



(D) $p=5$.

First order asymptotics

Let $L_{n,p}$ be a random measure having same law as $\frac{1}{n}\sum_{i=1}^{n}\delta_{x_i}$ under $\mathbb{P}_{n,p}$.

Lemma. There exist $B, c, C > 0$ such that, for $U := (-B, B)$,

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and Theorem A follows ($\langle \mu_p, x^2\rangle = \frac{p}{2p+4}$ using the density of μ_p).

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and Theorem A follows ($\langle\mu_p, x^2\rangle = \frac{p}{2p+4}$ using the density of μ_p).
- As for Theorem B, at this point we can only conclude that
$$\lim_{n\to\infty}\frac{I_q\big(B_E\big(S_p^n\big)\big)}{I_2\big(B_E\big(S_p^n\big)\big)} = 1.$$

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We know that $\langle L_{n,p}, x^2 \rangle \to \langle \mu_p, x^2 \rangle$ a.s. and in moments. What can we say about

$$F_{n,p} := n\Big(\langle L_{n,p}, x^2 \rangle - \langle \mu_p, x^2 \rangle\Big)?$$

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Theorem C. Suppose $E = \{T = T^*\}$ and $p \in (3, \infty)$. As $n \to \infty$, $F_{n,p}$ converges in law and in moments to some $\mathcal{N}(m_p, \frac{1}{4b})$ variable.

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We deduce that

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Theorem B follows by raising to $\frac{1}{q}$ and replacing $\operatorname{Var} F_{n,p}$ and $\langle \mu_p, x^2\rangle$.

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- Taylor-expand as $t \to 0$ ($n \to \infty$) and choose $\psi_p \in \mathcal{C}^3$ solution to

$$V_p'(x)\psi(x) - 2\int \frac{\psi(x) - \psi(y)}{x - y}\,\mu_p(\mathrm{d}y) = x^2 + c, \qquad x \in \mathbb{R}.$$

Thanks for your attention!