Renormalization groups, transport maps, and multiscale Bakry-Émery criteria

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Renormalization groups

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- Renormalization groups tackle this problem by reducing the number of degrees of freedom so that the number of "effective" degrees of freedom per correlation length is small.
- The process of elimination of the degrees of freedom is gradual: going from one step to the next, only a few degrees of freedom need to be considered which simplifies the analysis.

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- Quantum field theories: Behavior at infrared and ultraviolet limits, asymptotic freedom (quantum chromodynamics), asymptotic safety (quantum gravity)....

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- **1970s**: Wilson developed (in the context of statistical field theories) *exact* (non-perturbative) renormalization (semi)groups.
- **1980s**: We will use a version of an exact renormalization (semi)group due to Polchinski.

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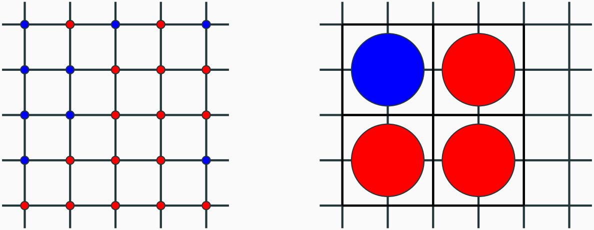
- Interacting field: A measure $\nu^{\epsilon,L}$ of the form $d\nu^{\epsilon,L} = e^{-V_0} d\gamma^{\epsilon,L}$.
- Continuum limits: $L \uparrow \infty$ (infrared—long distance scale) and $\epsilon \downarrow 0$ (ultraviolet—short distance scale).

Kadanoff's block spin picture



φ $\qquad\qquad$ $\tilde{\varphi}$

$$\text{Partition function} = Z = \sum_{\varphi} e^{-V_0(\varphi)} d\gamma^{\epsilon,L}(\varphi) = \sum_{\tilde{\varphi}} e^{-\tilde{V}_0(\tilde{\varphi})} d\widetilde{\gamma^{\epsilon,L}}(\tilde{\varphi})$$

$$e^{-\tilde{V}_0(\tilde{\varphi})} = \sum_{\varphi} 1_{\{\varphi \,\in\, \text{block } \tilde{\varphi}\}}\; e^{-V_0(\varphi)}$$

Intuition (wrong): $\tilde{V}_0(\tilde{\varphi})$ is of the same form as $V_0(\varphi)$ with different coupling constants.

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Infinitesimal transformation on **infinite** couplings with **soft** cutoff:

$$e^{-V_t(\varphi)} = \int_{\zeta} e^{-V_0(\zeta)} d\gamma_{C_t}(\zeta - \varphi) = \mathbb{E}_{C_t}[e^{-V_0(\varphi+\zeta)}].$$

The Langevin transport map

Langevin dynamics

Let $(\Phi_t)_{t\geq 0}$ be the Langevin dynamics:

$$d\Phi_t = \nabla \log\left(\frac{d\mu}{d\varphi}\right)(\Phi_t)dt + \sqrt{2}dB_t, \quad \Phi_0 \sim \nu^{\epsilon,L},$$

with $(B_t)_{t\geq 0}$ a Brownian motion in $\mathbb{R}^{\Lambda_{\epsilon,L}}$.

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Let (U_t) be the Langevin semigroup: $U_t \eta(\varphi) = \mathbb{E}[\eta(\Phi_t) | \Phi_0 = \varphi]$.

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Let $\rho_t := \text{Law}(\Phi_t) = U_t\left(\frac{d\nu^{\epsilon,L}}{d\mu}\right)d\mu$ so the path of measures $(\rho_t)_{t\geq 0}$ interpolates between $\rho_0 = \nu^{\epsilon,L}$ to $\rho_\infty = \mu$.

The continuity equation

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The Langevin path $(\rho_t)_{t\geq 0}$ satisfies the continuity equation

$$\partial_t \rho_t + \nabla(\nabla u_t \rho_t) = 0,$$

where

$$\nabla u_t(\varphi) = -\nabla \log \left(\frac{d\rho_t}{d\mu} \right)(\varphi) = -\nabla \log U_t \left(\frac{d\nu^{\epsilon,L}}{d\mu} \right)(\varphi),$$

because $\partial_t U_t \eta = \Delta U_t \eta + \left(\nabla U_t \eta, \nabla \log \left(\frac{d\mu}{d\varphi} \right) \right)$.

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Define the family of diffeomorphisms $S_t : \mathbb{R}^{\Lambda_{\epsilon,L}} \to \mathbb{R}^{\Lambda_{\epsilon,L}}$ by

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S_t transports $\nu^{\epsilon,L} = \rho_0$ to ρ_t and $T_t := S_t^{-1}$ transports ρ_t to $\rho_0 = \nu^{\epsilon,L}$. The **Langevin transport map** is

$$T_{\mathrm{LVN}} := \lim_{t\to\infty} T_t \quad \text{transports } \mu = \rho_\infty \text{ to } \rho_0 = \nu^{\epsilon,L}.$$

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- Kim & Milman were the first to show that T_{LVN} enjoys Lipschitz properties.
- Further Lipschitz properties of T_{LVN} were proven by Klartag & Putterman, Mikulincer & S., and Neeman.
- Tanana showed that, in general, T_{LVN} is different than the optimal transport map.

Transport of functional inequalities

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Suppose μ satisfies a Poincaré inequality with constant C_μ:

$$\mathrm{Var}_\mu(F) \leq C_\mu \mathbb{E}_\mu \left[|\nabla F|^2 \right] \quad \text{for all test-function } F : \mathbb{R}^{\Lambda_{\epsilon,L}} \to \mathbb{R}.$$

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$$\begin{aligned} &\mathrm{Var}_{\nu^{\epsilon,L}}[F] = \mathrm{Var}_\mu(F \circ T) \leq C_\mu \mathbb{E}_\mu \left[|\nabla (F \circ T)|^2 \right] \\ &\leq C_\mu \mathbb{E}_\mu \left[|DT|^2 |\nabla F(T)|^2 \right] \leq L^2 C_\mu \mathbb{E}_\mu \left[|\nabla F(T)|^2 \right] \\ &= L^2 C_\mu \mathbb{E}_{\nu^{\epsilon,L}} \left[|\nabla F|^2 \right]. \end{aligned}$$

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\end{aligned}$$

The **transport method** is very general and allows the transfer of numerous functional inequalities.

Exact renormalization groups and transportation of measures

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We choose $\dot{C}_t = e^{-t\frac{A_\epsilon}{2}}$ and $\tau = \infty$.

The multiscale Bakry-Émery criterion

Definition. A model $\nu^{\epsilon,L}$ satisfies the **multiscale Bakry-Émery criterion** if the exist real number $\dot{\lambda}_t$ (possibly negative) such that, for any $t \geq 0$ and $\varphi \in \mathbb{R}^{\Lambda_{\epsilon,L}}$,

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This criterion (and versions thereof) was used by Bauerschmidt & Bodineau and Bauerschmidt & Dagallier to prove Poincaré and log-Sobolev inequalities for various field theories $\nu^{\epsilon,L}$. The proofs follow the Bakry-Émery theory but using the **Polchinski semigroup** rather than the **Langevin semigroup**.

New perspective on Polchinski's exact renormalization group

Theorem [S.] Suppose a smooth[1] model $\nu^{\epsilon,L}$ satisfies the multiscale Bakry-Émery criterion. Then, the Langevin transport map T_{LVN}, which pushes forward $\gamma^{\epsilon,L}$ to $\nu^{\epsilon,L}$, is $\exp\left(\frac{1}{2}\int_0^\infty \dot{\lambda}_t ds\right)$-Lipschitz.

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Remark 2. Our new perspective on exact renormalization groups views them as transporting the free field theories (Gaussian) $\gamma^{\epsilon,L}$ to interacting field theories $\nu^{\epsilon,L}$ in *non-perturbative* way. When the transport maps are *Lipschitz*, the interacting field theories $\nu^{\epsilon,L}$ can be controlled.

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The (two-dimensional massive) sine-Gordon model

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Applying the Theorem yields many functional inequalities which previously were not known.

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$$\nabla^2 u_t(\varphi) = -e^{-t\frac{A_\epsilon}{2}} \nabla^2 V_t(\varphi) e^{-t\frac{A_\epsilon}{2}}.$$

Hence, if $\nu^{\epsilon,L}$ satisfies the multiscale Bakry-Émery criterion, then

$$-\nabla^2 u_t(\varphi) \succeq \dot{\lambda}_t \,\mathrm{Id}.$$

Using

$$\partial_t \nabla S_t(\varphi) = -\nabla^2 u_t(\varphi) \nabla S_t(\varphi),$$

completes the proof.

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The **Polchinski semigroup** $(P_{s,t})_{0\leq s\leq t}$ is a time-inhomogeneous semigroup given by

$$P_{s,t}F(\varphi) := \mathbb{E}[F(\Psi_s)|\Psi_t = \varphi],$$

where $\Psi_t := \tilde{\Phi}_{\tau-t}$.

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- The process (Ψ_t) is obtained by taking the martingale $\left(\int_0^t \dot{C}_r^{1/2} dB_r\right)_t$, which satisfies $\int_0^\tau \dot{C}_r^{1/2} dB_r \sim \gamma^{\epsilon,L}$, and conditioning it so that at time τ is distributed like $\nu^{\epsilon,L}$.

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- When $\tau = 1$ and $\dot{C}_r = \mathrm{Id}$ for all r, Mikulincer & S. constructed the **Brownian transport map** based on the process (Ψ_t). Using, implicitly, a multiscale Bakry-Émery criterion, we showed that the Brownian transport map satisfies Lipschitz properties.

Future work

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Thank You